Equivalence of two mathematical models of injecting drug use, using Cauchy's integral theorem

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Abstract

In this paper we derive two mathematical expressions for the expected number of people who are susceptible to acquiring an infection from a contaminated syringe in a setting of a group of people who share the same syringe, some of whom may be infected with a virus. We then prove the equality of the two expressions by using Cauchy's integral theorem. Interestingly, this result is also an implication of a special case of a Chu–Vandermonde summation and hypergeometric series. This proof will assist the field of mathematical modelling of disease transmission among injecting drug users. The method can also be applied to other problems involving sequential ordering of entities.

Introduction

Injecting drug users (IDUs) who share injecting equipment with other people are an important group at risk of acquiring infectious viral agents [1]. For example, HIV epidemics in some Asian countries have predominantly been driven by IDUs [2]. At the same time, the prevalence of hepatitis C among IDUs in Australia is very high (at 50–70%), but fortunately the prevalence of HIV in Australian IDUs is relatively low (at $< 1\%$) [3], [4]. Viral transmissions can occur when an individual infected with a virus uses a needle and syringe, the needle becomes contaminated, and then it is used by someone else. The incidence of infections can be magnified when IDUs form groups where a single needle and syringe may be used by many people.

Various mathematical models have been developed in an attempt to describe the population-level incidence of blood-borne infections among IDUs and to suggest public health policies to control epidemics (e.g. [5], [6], [7]). The complexities of social behaviour and viral transmission have demanded increasingly more complicated models. This presents two problems. First, more complicated models require more time for numerical computation and analysis and are prone to more

Received 25 November 2008; accepted for publication 16 April 2010.
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mistakes. Second, two models of the same virus and situation may differ significantly in description and as such cannot be meaningfully compared with each other.

Here, mathematical models are developed for a group of IDUs sharing the same needle. Two methods are suggested for calculating the average exposure of an uninfected person. The first method considers average intervals of people between infected IDUs and the second method considers the possible permutations of the ordering of uninfected IDUs before the first infected IDU uses the injecting equipment.

Method 1

If the number of people in an injecting sharing group is m and there are r people in the group who are infected with a given blood-borne virus then $m-r$ are not infected. Assuming that all group members inject with the same injecting equipment in random order, the expected number of uninfected IDUs between two infected individuals injecting is $(m-r)/(r+1)$. This is also the expected number of uninfected IDUs who use the needle before the first infected person uses and potentially contaminates it. Consequently, the expected number of people in the group who are susceptible to acquiring the virus is the number of uninfected people who use the needle after an infected person, that is,

$$m - r - \frac{m-r}{r+1} = \frac{r(m-r)}{r+1} \,. \tag{1}$$

Method 2

The probability of s uninfected people in the group (of size m) using a syringe before the first (of r) infected person uses the syringe is given by

$$\begin{aligned} \Pr(s) = {} & \overbrace{\left(\frac{m-r}{m}\right)}^{\text{1st person uninfected}} \quad \overbrace{\left(\frac{m-r-1}{m-1}\right)}^{\text{2nd person uninfected}} \\ & \cdots \overbrace{\left(\frac{m-r-(s-1)}{m-(s-1)}\right)}^{s\text{th person uninfected}} \quad \overbrace{\left(\frac{r}{m-s}\right)}^{s+1\text{th person infected}} \\ = {} & \frac{(m-r)!}{(m-r-s)!}\frac{(m-s)!}{m!}\left(\frac{r}{m-s}\right). \end{aligned} \tag{2}$$

The expected number of uninfected people to use the syringe before the first infected person is $\sum_{s=0}^{m-r} \Pr(s)s$. Therefore, the expected number of susceptible people in the sharing group is

$$m - r - \sum_{s=0}^{m-r} \frac{(m-r)!}{(m-r-s)!}\frac{(m-s)!}{m!}\left(\frac{rs}{m-s}\right). \tag{3}$$

Numerical investigation indicates that expression (1) is equivalent to expression (3). Indeed, the equality of (1) and (3) can be shown by a special case of Chu–Vandermonde summation of a hypergeometric series [8]. Hypergeometric series were studied extensively in the nineteenth and early twentieth centuries, from the time of Gauss onwards [9], [10]. However, we present below an alternative proof involving Cauchy's integral theorem. We believe that our method may be useful for other problems involving sequential ordering, especially in the case of mathematical models of epidemics.

Proof. Our aim is to prove the following equality,

$$m - r - \sum_{s=0}^{m-r} \frac{(m-r)!}{(m-r-s)!} \frac{(m-s)!}{m!} \left(\frac{rs}{m-s} \right) = \frac{rm - r^2}{r+1}. \tag{4}$$

Since $r \neq m$, this equality can be seen to be equivalent to

$$1 - \frac{r(m-r-1)!}{m!} \sum_{s=0}^{m-r} \frac{(m-s-1)!}{(m-s-r)!}\, s = \frac{r}{r+1}.$$

Rearranging the equation so that the sum is the subject and dropping the first term in the series (which is 0), we obtain the following equality, which is to be proved:

$$\sum_{s=1}^{m-r} \frac{(m-s-1)!}{(m-s-r)!}\, s = \frac{1}{r(r+1)} \frac{m!}{(m-r-1)!} \tag{5}$$

Now consider the following binomial and its factorisation,

$$x^m - y^m = (x-y) \sum_{s=0}^{m-1} x^s y^{m-1-s}.$$

Partial differentiation with respect to x gives the following result,

$$mx^{m-1} = \sum_{s=0}^{m-1} x^s y^{m-1-s} + (x-y) \sum_{s=1}^{m-1} s x^{s-1} y^{m-1-s}.$$

Letting x=1 and rearranging yields

$$m - \frac{1-y^m}{1-y} = (1-y) \sum_{s=1}^{m-1} s y^{m-1-s}.$$

So,

$$\sum_{s=1}^{m-1} s y^{m-1-s} = \frac{m(1-y) - (1-y^m)}{(1-y)^2}.$$

We now define the following two functions,

$$f(y) := \sum_{s=1}^{m-1} s y^{m-1-s}$$

$$g(y) := \frac{m(1-y) - 1 + y^m}{(1-y)^2}$$

with the result that $f(y) = g(y)$ for all $y \neq 1$. The derivatives of f are, trivially,

$$f^{(r-1)}(y) = \sum_{s=1}^{m-r} \frac{(m-s-1)!}{(m-s-r)!}\, s y^{m-s-r}.$$

Hence,

$$f^{(r-1)}(1) = \sum_{s=1}^{m-r} \frac{(m-s-1)!}{(m-s-r)!}\, s, \tag{6}$$

which is equal to the left hand side of (5). Since $f(y)$ is a polynomial, it is analytic on the entire complex plane and by Cauchy's integral theorem,

$$f^{(r-1)}(1) = \frac{(r-1)!}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z-1)^r}\, \mathrm{d}z$$

for any curve, $\mathcal{C}$, containing the point $z_0 = 1$. Let $\mathcal{C}$ be the circle in $\mathbb{C}$ of radius 1 centred about the point 1. So,

$$z - 1 = \mathrm{e}^{\mathrm{i}\theta}, \quad \theta \in [0, 2\pi].$$

Hence the integral becomes

$$f^{(r-1)}(1) = \frac{(r-1)!}{2\pi \mathrm{i}} \int_0^{2\pi} f(1+\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{i}\mathrm{e}^{-\mathrm{i}(r-1)\theta}\, \mathrm{d}\theta.$$

Since the integral is not evaluated at the point $z = 1$, we can replace $f(z)$ with $g(z)$,

$$\begin{aligned}
f^{(r-1)}(1) &= \frac{(r-1)!}{2\pi \mathrm{i}} \int_0^{2\pi} g(1+\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{i}\mathrm{e}^{-\mathrm{i}(r-1)\theta}\, \mathrm{d}\theta \\
&= \frac{(r-1)!}{2\pi} \int_0^{2\pi} \frac{-m\mathrm{e}^{\mathrm{i}\theta} - 1 + (1+\mathrm{e}^{\mathrm{i}\theta})^m}{\mathrm{e}^{2\mathrm{i}\theta}}\, \mathrm{e}^{-\mathrm{i}(r-1)\theta}\, \mathrm{d}\theta \\
&= \frac{(r-1)!}{2\pi} \int_0^{2\pi} (m\mathrm{e}^{-\mathrm{i}r\theta} - \mathrm{e}^{-\mathrm{i}(r+1)\theta} + \mathrm{e}^{-\mathrm{i}(r+1)\theta}(1+\mathrm{e}^{\mathrm{i}\theta})^m)\, \mathrm{d}\theta \\
&= \frac{(r-1)!}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}(r+1)\theta} \sum_{k=2}^{m} \binom{m}{k} \mathrm{e}^{\mathrm{i}k\theta}\, \mathrm{d}\theta \\
&= \frac{(r-1)!}{2\pi} \sum_{k=2}^{m} \binom{m}{k} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(k-r-1)\theta}\, \mathrm{d}\theta.
\end{aligned}$$

The only nonzero contribution is from the term where $k = r + 1$, and so

$$\begin{aligned}
f^{(r-1)}(1) &= \frac{(r-1)!}{2\pi} 2\pi \binom{m}{r+1} \\
&= \frac{(r-1)!\, m!}{(r+1)!\,(m-r-1)!} \\
&= \frac{1}{r(r+1)} \frac{m!}{(m-r-1)!}\,,
\end{aligned} \tag{7}$$

which is equivalent to the right-hand side of equation (5). □

Acknowledgements

We wish to thank Richard Gray and Matthew Law for their efforts towards this work. We also thank an anonymous reviewer for bringing the Chu-Vandermonde identity to our attention. This study was funded by the Australian Government Department of Health and Ageing. The views expressed in this publication do not necessarily represent the position of the Australian Government. NCHECR is affiliated with the Faculty of Medicine, University of New South Wales.

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