

Centroid of a quadrilateral

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Abstract

We give a simple rule-and-compasses construction for locating the centroid of a plane quadrilateral lamina, and we make some deductions about linear dependence in the plane.

Let $\Omega = PQRS$ be a convex quadrilateral, and Ω^* the lamina $\Omega \cup$ inside (Ω); its area A is of course the sum of areas of two triangles. We introduce the notation A_P for the area of the triangle *having vertex P and the opposite diagonal as its opposite side*, that is $\triangle PQS$; so that

$$A = A_P + A_R = A_Q + A_S. \quad (1)$$

Let an origin O be taken anywhere in the plane not collinear with any two of the vertices, and let $\mathbf{P}, \mathbf{Q}, \dots$ denote the position vectors of the points P, Q, \dots from O . Although the four vectors $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are linearly dependent, they form a convenient spanning set for the plane; any linear combination of them denotes a unique point in the plane, even though no point has a unique such representation. Consider the points (see Figure 1) and their vectors

$$\begin{aligned} \mathbf{M} &= \frac{1}{2}(\mathbf{P} + \mathbf{R}), & \mathbf{N} &= \frac{1}{2}(\mathbf{Q} + \mathbf{S}), \\ \mathbf{X} &= \frac{1}{4}(\mathbf{P} + \mathbf{Q} + \mathbf{R} + \mathbf{S}), & \mathbf{W} &= (A_P\mathbf{P} + A_Q\mathbf{Q} + A_R\mathbf{R} + A_S\mathbf{S})/2A. \end{aligned} \quad (2)$$

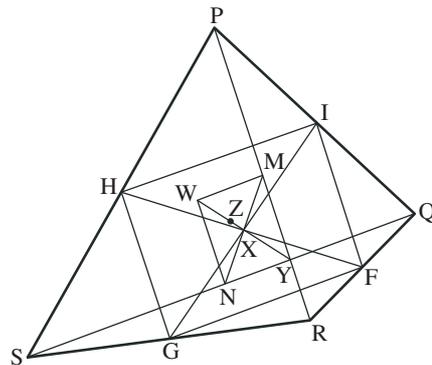


Figure 1.

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The points M and N are the midpoints of the diagonals of Ω ; X is the unweighted mean of the vertices, both of Ω and of Λ (the median parallelogram of Ω), and is also the point of intersection of the diagonals of Λ . (Λ is defined to be the parallelogram FGHI whose vertices are the midpoints of the sides of Ω , taken in order; see [1].) We define Y and Z presently.

The centroid of the lamina Ω^* can be calculated as the mean of the centroids of the two triangles making up the quadrilateral, say $\triangle PRQ$ and $\triangle PRS$, weighted according to the areas of these triangles. The centroid of a triangle is the common point of its medians, so the centroid of $\triangle PRQ$ is $\frac{1}{3}(\mathbf{P} + \mathbf{Q} + \mathbf{R}) = (2\mathbf{M} + \mathbf{Q})/3$; thus the centroid of Ω is

$$\begin{aligned}\mathbf{Z} &= (A_Q(\mathbf{P} + \mathbf{Q} + \mathbf{R})/3 + A_S(\mathbf{P} + \mathbf{R} + \mathbf{S})/3)/(A_Q + A_S) \\ &= 2\mathbf{M}/3 + (A_Q\mathbf{Q} + A_S\mathbf{S})/3A.\end{aligned}\quad (3)$$

Similarly, by regarding Ω as made up of $\triangle PQS$ and $\triangle RQS$, we get

$$\mathbf{Z} = 2\mathbf{N}/3 + (A_P\mathbf{P} + A_R\mathbf{R})/3A. \quad (4)$$

Averaging (3) and (4) gives

$$\mathbf{Z} = \frac{1}{3}(\mathbf{M} + \mathbf{N}) + \frac{1}{3}\mathbf{W} = \frac{1}{3}(2\mathbf{X} + \mathbf{W}). \quad (5)$$

Lemma 1. $A_R\mathbf{P} + A_P\mathbf{R} = A_S\mathbf{Q} + A_Q\mathbf{S} = A\mathbf{Y}$, where Y is the point of intersection of the diagonals of Ω .

Proof. Let Y be the point of intersection of PR and QS. Clearly $\mathbf{Y} = (|YR|\mathbf{P} + |YP|\mathbf{R})/|PR| = (|\triangle RQS|\mathbf{P} + |\triangle PQS|\mathbf{R})/A$, since the areas of these two triangles on the same base QS are to one another as their vertical heights, and hence as the lengths |YR|, |YP|. That is, $A\mathbf{Y} = A_R\mathbf{P} + A_P\mathbf{R}$. The second equation is proved in the same way. \square

From these formulae we easily deduce the equality of the vectors \overrightarrow{WM} and \overrightarrow{NY} , proving that $\Psi := MYNW$ is a parallelogram. Also X is the midpoint of WY. Then (5) gives the location of the centroid on XW. We have established the following Construction.

Construction. To find the centroid Z of the quadrilateral lamina Ω^* :

- (1) draw the diagonals of Ω and locate their point of intersection Y and their midpoints M, N;
- (2) find the midpoint X of MN;
- (3) find W, as the fourth vertex of the parallelogram MYNW;

then Z is the point of trisection of XW which is the closer to X.

All these steps are effected by simple use of straight-edge and compasses.

The parallelogram Ψ has sides parallel to those of Λ . Also Ψ and Λ have the same centre, but although Ψ is symmetrically placed inside Λ , it is in general not proportional to Λ .

Dart. The above discussion deals with the case when Ω is convex. Suppose instead that Ω is a dart, with its concavity at S and point at Q. See Figure 2. There are some changes to the signs of terms, but the theory is similar. Instead of (1) we have $A = A_P + A_R = A_Q - A_S$, and Y is the point of intersection of QS produced and PR. Thus in place of Lemma 1 we have $A_R\mathbf{P} + A_P\mathbf{R} = -A_S\mathbf{Q} + A_Q\mathbf{S} = A\mathbf{Y}$, and the new definition of W is

$$\mathbf{W} = (A_P\mathbf{P} + A_Q\mathbf{Q} + A_R\mathbf{R} - A_S\mathbf{S})/2A.$$

With these changes (5) holds, and the same construction applies.

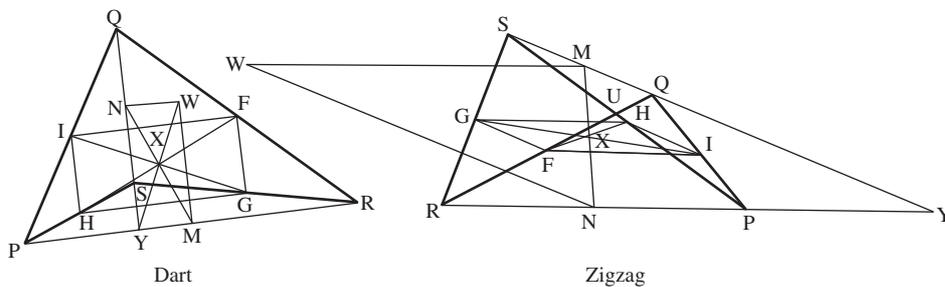


Figure 2.

Zigzag. There is now a new point U, the intersection of two opposite sides, say PS and QR. The natural adaption of the preceding theory takes A to be the *difference* of the areas of the two components of inside(Ω). As before, $A_P = \text{area}(\triangle PQS)$, etc. Assume without loss of generality that RP produced meets SQ produced in Y. Then $A = A_R - A_P = A_S - A_Q$, and now

$$A\mathbf{Y} = A_R\mathbf{P} - A_P\mathbf{R} = -A_S\mathbf{Q} + A_Q\mathbf{S},$$

$$\mathbf{W} = (-A_P\mathbf{P} - A_Q\mathbf{Q} + A_R\mathbf{R} + A_S\mathbf{S})/2A.$$

With these changes (5) holds, and the same construction applies.

Linear dependence in R^2

There must exist two independent linear relations among the four vectors $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$. Confining attention to the convex case, we get one relation from Lemma 1, namely

$$A_R\mathbf{P} - A_S\mathbf{Q} + A_P\mathbf{R} - A_Q\mathbf{S} = \mathbf{0}, \tag{6}$$

in which moreover the coefficients are all independent of the choice of origin (from their definition as areas), and also their sum is zero. One might hope that a second dependence relation could be found by using one of the zigzags PRQS or PQSR and their equivalent forms of (6), but this leads back to (6).

Let us call a nontrivial linear dependence relation *zero-sum* when the sum of the coefficients is zero; (6) is one such. We can easily prove Lemma 2.

Lemma 2. Given $\Omega = PQRS$, the following properties are equivalent, for any tuple $(\alpha, \beta, \gamma, \delta)$ of reals:

- (a) There exists an origin O such that $\alpha\mathbf{P} + \beta\mathbf{Q} + \gamma\mathbf{R} + \delta\mathbf{S} = \mathbf{0}$ holds as a zero-sum dependence relation on the position vectors from O , that is, where $\mathbf{P} = \overrightarrow{OP}, \dots$
- (b) For every choice of origin O' and position vectors $\mathbf{P}' = \overrightarrow{O'P}, \dots$, the linear dependence relation $\alpha\mathbf{P}' + \beta\mathbf{Q}' + \gamma\mathbf{R}' + \delta\mathbf{S}' = \mathbf{0}'$ holds.

Thus the property of being a zero-sum relation is equivalent to the relation being independent of the choice of origin in the plane. Furthermore, we have Lemma 3.

Lemma 3. If there holds a second zero-sum linear dependence relation for Ω not equivalent to (6), then every linear dependence relation for Ω is zero-sum.

Proof. Any third linear dependence relation must be a linear combination of the given two, and therefore zero-sum. \square

Take the origin O to be the centroid; then $\mathbf{Z} = \mathbf{0}$ and (5) and (2) give

$$(A + A_P)\mathbf{P} + (A + A_Q)\mathbf{Q} + (A + A_R)\mathbf{R} + (A + A_S)\mathbf{S} = \mathbf{0}.$$

This linear relation is certainly not zero-sum. By Lemma 3 we deduce Theorem 1.

Theorem 1. *Relation (6) is (to within a constant multiple) the sole zero-sum linear dependence relation for the convex quadrilateral Ω .*

Similar results hold for darts and zigzags. The theorem describes a general property of four position vectors in the plane, no two of them linearly dependent.

References

- [1] Miller, J.B. (2007). Plane quadrilaterals. *Gaz. Aust. Math. Soc.* **34**, 103–111.