Puzzle corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner Number 17. Each issue will include a handful of fun, yet intriguing, puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes SA 5095.

The deadline for submission of solutions for Puzzle Corner 17 is 1 July 2010. The solutions to Puzzle Corner 17 will appear in Puzzle Corner 19 in the September 2010 issue of the *Gazette*.

Railway repairs

In the year 2020, there is a railway service connecting Melbourne and Hobart. Trains run at full speed except for two railway segments, where poor track conditions force them to slow down. If any one of those two segments were repaired, the average speed of a train between Melbourne and Hobart would increase by a third. How much would the average speed between Melbourne and Hobart increase if both segments were repaired?

Circles and tangents

Let there be a unit circle inscribed in an equilateral triangle. A smaller circle is tangent to two sides of the triangle as well as the first circle. What is the radius of the smaller circle?

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Peaceful pairing

There are n red points and n blue points in the plane, such that no three points are collinear. Is it always possible to divide them into n red–blue pairs, such that the line segments from each red point to its corresponding blue point do not cross?

Chessboard parity

In a certain chess position on a regular chessboard, each horizontal row and each vertical column has an even number of pieces. Prove that the number of pieces on black squares is also an even number.

Bonus: If each horizontal row and each vertical column has an odd number of pieces instead, can the number of pieces on black squares ever be an odd number?

Tip the balance

A balance scale sits on the teacher's table, currently tipped to the right. There is a set of weights on the scales, and on each weight is the name of at least one student. The teacher chooses a set of students to enter the classroom. One by one, each chosen student will move each weight carrying his or her name to the opposite side. Prove that the teacher can always choose a set of students to tip the scales to the left.

Digital dilemmas

1. Take all the positive integers with 10 or fewer digits, and divide them into two sets. The set A contains those that can be written as the product of two integers with five or fewer digits, while the set B contains the rest. Which of the sets A and B contains more elements?
2. Take all the positive integers with 10 or fewer digits, and divide them into two sets. The set A contains those with odd digit sums, while the set B contains those with even digit sums. Now let the sum of the fifth powers of the numbers in A be x, and let the sum of the fifth powers of the numbers in B be y. Which of x and y is bigger?

Solutions to Puzzle Corner 15

The $50 book voucher for the best submission to Puzzle Corner 15 is awarded to David Angell. Many thanks to Norman Do for collecting and compiling the following solutions.

Dollars and cents

Solution by: David Angell

We write all monetary amounts in terms of cents. The key fact is the observation that there exists a coin of value c if and only if there exists one of value $100/c$. Now suppose that B coins of values $c_1, c_2, \ldots, c_B$ have sum A. Consider c_1 coins of value $100/c_1$, c_2 coins of value $100/c_2$, and so on, up to c_B coins of value $100/c_B$. Then there are

$$c_1 + c_2 + \cdots + c_B = A$$

coins with a total value of

$$c_1 \times 100/c_1 + c_2 \times 100/c_2 + \cdots + c_B \times 100/c_B = 100B.$$

Fair game

Solution by: Samuel Mueller

Albert and Betty's coin tosses can be considered as independent and identically distributed Bernoulli random variables $A_1, A_2, \ldots, A_{11}, B_1, B_2, \ldots, B_{10}$, each with binomial distribution $B(1, \frac{1}{2})$. If we let $A = A_1 + A_2 + \cdots + A_{10}$ and $B = B_1 + B_2 + \cdots + B_{10}$, then

$$P(A + A_{11} > B) = P(A > B) + P(A = B \text{ and } A_{11} = 1).$$

However, note that $P(A > B) = \frac{1}{2}P(A \neq B)$ while $P(A = B \text{ and } A_{11} = 1) = \frac{1}{2}P(A = B)$. Therefore, we conclude that

$$P(A + A_{11} > B) = \frac{1}{2}P(A \neq B) + \frac{1}{2}P(A = B) = \frac{1}{2}.$$

Evaluation

Solution by: Amy Glen

Since $f(10^6) = f(0) = 0$, the sum we wish to evaluate is

$$S = 2^{f(0)} + 2^{f(1)} + 2^{f(2)} + \cdots + 2^{f(999\,999)}.$$

Let us define F_k to be the number of $0 \leq n \leq 999\,999$ such that $f(n) = k$. It is easy to see that $0 \leq f(n) \leq 6$ for all $0 \leq n \leq 999\,999$, so we have

$$S = \sum_{k=0}^{6} F_k \times 2^k.$$

Consider a number $0 \leq n \leq 999\,999$ as a 6-digit string by padding it with zeroes. For example, we consider the number 1729 as the 6-digit string 001729. Therefore, F_k is equal to the number of 6-digit strings which contain exactly k digits equal to 9. This number is $\binom{6}{k}9^{6-k}$ since there are $\binom{6}{k}$ choices for the positions of the digits equal to 9 and also 9 choices for each of the $6 - k$ remaining positions. So by the binomial theorem, we have

$$S = \sum_{k=0}^{6} \binom{6}{k} 9^{6-k} \times 2^k = (9+2)^6 = 11^6.$$

Congruent cakes

Solution by: Rick Mabry

Consider the points on the sides of a triangle which divide the sides into k equal segments. If we connect these points with lines parallel to the sides, then we have divided the triangle into k^2 congruent triangles, all similar to the original.

Now consider a right-angled triangle with legs of integral lengths $m < n$. The altitude to the hypotenuse divides the triangle into two right-angled triangles which are both similar to the original. Divide the smaller one into m^2 triangles and the larger one into n^2 triangles using the method above. This creates $m^2 + n^2$ congruent triangles which are all similar to the original. Since $28^2 + 35^2 = 2009$, it is possible to cut a triangular cake in the shape of a right-angled triangle with legs of lengths 28 and 35 into 2009 congruent triangular pieces.

Comment. Rick has also provided the following more facetious solution, which uses the fact that the slices must actually be three-dimensional. Assuming the cake to be a triangular slab of uniform thickness, we can make 2009 congruent slices with thickness equal to $\frac{1}{2009}$ times the thickness of the cake.

Matrix mayhem

Solution by: Reiner Pope

1. We simply choose a row or column whose sum is strictly negative and negate each number in it. We will prove that repeating this process must terminate when the sums of the numbers in each row and column are all non-negative. Note that if we negate the numbers in such a row or column, then the sum of all elements in the matrix strictly increases. However, there are at most $2^{m \times n}$ possible sums of all the elements in the matrix, one for every possible choice of signs for the individual elements. Since this is finite, the process must terminate, in which case the sums of the numbers in each row and column are all non-negative.
2. The problem is trivially true if there exists a column which contains the same number N times. In that case, we can simply delete this column and the rows of the resulting matrix are still pairwise distinct.
   Now suppose that every column contains at least two distinct numbers. Let $f(k)$ denote the number of distinct rows in the matrix formed by taking the leftmost k columns. Clearly, f is non-decreasing, $f(1) \geq 2$ and $f(N) = N$. Therefore, there must exist some $1 < M \leq N$ for which $f(M-1) = f(M)$. We claim that if the Mth column is removed, then the rows of the resulting matrix are still pairwise distinct.

To prove this claim, suppose that $g(a,b)$ is the index of the leftmost column for which rows a and b have distinct entries. Since $f(M-1) = f(M)$, we cannot have $g(a,b) = M$. Since a and b are arbitrary, it follows that the rows of the resulting matrix are still pairwise distinct.

3. Consider the monic polynomial of degree 100

$$P(x) = -1 + \prod_{i=1}^{100}(x + a_i).$$

We are given that $P(b_j) = 0$ for $1 \leq j \leq 100$, so $b_1, b_2, \ldots, b_{100}$ must be the distinct roots of $P(x)$. Thus, we have the equality

$$P(x) = \prod_{j=1}^{100}(x - b_j),$$

and the product of the numbers along row k is

$$\prod_{j=1}^{100}(a_k + b_j) = \prod_{j=1}^{100}(-a_k - b_j) = P(-a_k) = -1 + \prod_{i=1}^{100}(-a_k + a_i) = -1.$$

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.