Mathematics Olympiads: Good Problems Appeal

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About once in three years the Senior Problems Committee of the Australian Mathematical Olympiad Committee (AMOC) turns to our mathematical community with an appeal for problem donations that can be used in national, regional and international senior secondary school mathematics competitions. The latest appeal [2] provided examples of competition problems that had been set for various contests in Australia and in the Asia-Pacific region between 2004 and early 2007. The present article is to repeat this exercise with problems from competitions held between 2008 and early 2010. Problems chosen for these competitions are from 'pre-calculus' areas such as geometry (with a strong preference for 'Euclidean' geometry), number theory, algebra and combinatorics.

1. The AMOC Senior Contest is held in August of each year. About 100 students, most of them in Year 11, are given five problems and four hours to solve them. The following problem, Question 2 of the 2008 contest, turned out to be relatively easy:

> Let ABC be an acute-angled triangle, and let D be the point on AB (extended if necessary) such that AB and CD are perpendicular. Further, let t_A and t_B be the tangents to the circumcircle of ABC through A and B, respectively, and let E and F be the points on t_A and t_B, respectively, such that CE is perpendicular to t_A and CF is perpendicular to t_B. Prove that $CD/CE = CF/CD$.

2. The Australian Mathematical Olympiad (AMO) is a two-day event in February with about 100 participants. On each day, students are given a four-hour paper containing four problems. The following three problems from this competition had a 'filtering' effect in as much as they turned out to be not too easy for most students while the majority of our top contestants encountered minor difficulties.

Question 7 of the 2008 contest, proposed by Bolis Basit, Melbourne:

> Let $A_1A_2A_3$ and $B_1B_2B_3$ be triangles. If $p = A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1$ and $q = A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 + A_3B_1 + A_3B_2 + A_3B_3$, prove that $3p \leq 4q$.

Question 7 of the 2009 contest, proposed by Angelo Di Pasquale, Melbourne:

> Let I be the incentre of a triangle ABC in which $AC \neq BC$. Let Γ be the circle passing through A, I and B. Suppose Γ intersects the line AC at A and X and intersects the line BC at B and Y. Show that $AX = BY$.

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Question 6 of the 2010 contest, proposed by Joseph Kupka, Melbourne:

> Prove that
> $$\sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}+\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}}$$
> $$=\sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}+\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}\,.$$

3. Question 7, proposed by Ian Wanless, Melbourne, turned out to be really hard for contestants at the 2010 AMO and contributed much to the selection of students that are to represent Australia at the international level:

> Let a, b, c, d be integers satisfying $0 < a < b < c < d < 2010$. Prove that there exists an integer e satisfying: (i) $0 < e < 2010$, (ii) e is a divisor of 2010, and (iii) no two of a, b, c, d give the same remainder when divided by e.

4. The Asian Pacific Mathematics Olympiad (APMO) takes place in March. The contest is a four-hour event with five problems to be solved. About 30 countries now take part in the APMO. Usually, 25 to 30 Australian students are invited to participate in this competition. Considered hardest on the 2009 APMO paper by the APMO Problems Committee, which includes mathematicians from three different countries, was this problem submitted by Ivan Guo, Sydney:

> Larry and Rob are two robots travelling in a car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm. Larry makes a 90° left turn after every l kilometres and Rob makes a 90° right turn after every r kilometres driving from the start, where l and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and that the car can move in any direction. Let the car start from Argovia facing towards Zillis. For which choices of the pair (l, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

5. The International Mathematical Olympiad (IMO), initiated in 1959 with eight and now attended by more than 100 countries, is the top secondary-school competition globally.

Question 1 of the 2009 contest, deemed 'easy' by the international jury, had been submitted to the AMOC Senior Problems Committee by Ross Atkins, Canberra, went through several formulations and appeared on the IMO paper in the following form:

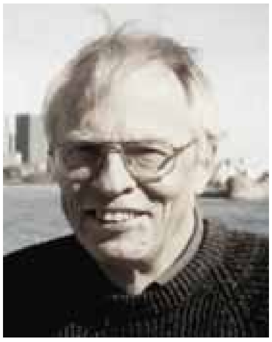
> Let n be a positive integer and let $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

The complete set of AMO problems and solutions covering the period 1979–1995 can be looked up in [4], whereas the problems and solutions of all APMOs between 1989 and 2000 have appeared in [3]. Furthermore, the problems, including solutions and statistics, of each year's AMOC Senior Contest, the AMO, the APMO, the International Mathematical Olympiad and some intermediate secondary school mathematics competitions are available in the AMOC's year books ([1]).

Please let me have your problem donation(s). As always, credit to the donor of successful problems will be given in [1].

References

[1] Brown, P.J., Di Pasquale, A. and McAvaney K.L. (appears annually). *Mathematics Contests: The Australian Scene*.
[2] Lausch, H. (2007). An Olympiad problem appeal. *Gaz. Aust. Math. Soc.* **34,** 90–91.
[3] Lausch, H. and Bosch Giral, C. (2000). *Asian Pacific Mathematics Olympiads 1989–2000*. Australian Mathematics Trust, Canberra, 2000.
[4] Lausch, H. and Taylor, P. (1997). *Australian Mathematical Olympiads 1979–1995*. Australian Mathematics Trust, Canberra, 1997.



Hans Lausch grew up as a research fellow in Bernhard Neumann's department at the ANU. In 1972 he joined Monash University. He has published in universal algebra, group theory and 18th-century mathematics history. From 1980 to 1982 he was the editor of Series A of the *Journal of the Australian Mathematical Society.* As the chairman of the Problems Committee of AMOC, the Australian Mathematical Olympiad Committee, he is looking after the collection, creation and development of contest and competition problems and therefore is grateful for problem donations. He represents Australia at the Asian Pacific Mathematics Olympiad.