Some results in the study of the continuous real functions

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Abstract

In this paper we consider two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, which satisfy a condition of the type $f(I) \cap g(J) \neq \varnothing$ where $I, J \subset \mathbb{R}$ are nonempty open intervals. We study some properties of the functions f and g about their continuity.

Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two real functions of one variable. In the first part of this paper we consider that the functions f and g satisfy the condition $f(I) \cap g(J) \neq \varnothing$ for all $I, J \subset \mathbb{R}$ nonempty open intervals. We have the following result:

Theorem 1. *If $\lim_{x\to\infty} g(x) = \infty$, then f is discontinuous at every point $x \in \mathbb{R}$.*

Proof. Let $x \in \mathbb{R}$ and n be a natural number, $n \geq 1$. Let $I_n = (x - \frac{1}{n}, x + \frac{1}{n})$ and $J_n = (n, \infty)$. We have $f(I_n) \cap g(J_n) \neq \varnothing$. It results that there exists $y_n \in f(I_n) \cap g(J_n)$. Consequently $y_n = f(u_n) = g(v_n)$ where $u_n \in I_n$ and $v_n \in J_n$. So we have obtained two sequences $(u_n)_n, (v_n)_n \subset \mathbb{R}$ with the properties $u_n \to x$ and $v_n \to \infty$. From $\lim_{x\to\infty} g(x) = \infty$, we obtain $y_n = g(v_n) \to \infty$. If f is continuous at the point x, then we have $y_n = f(u_n) \to f(x)$ and this is contradictory to the fact that $y_n \to \infty$. Thus the proof of Theorem 1 is complete. $\square$

We can observe that we obtain the same conclusion as in Theorem 1 if we replace the hypothesis $\lim_{x\to\infty} g(x) = \infty$ with one of the following:

$$\lim_{x\to\infty} g(x) = -\infty, \quad \lim_{x\to-\infty} g(x) = \infty \quad \text{or} \quad \lim_{x\to-\infty} g(x) = -\infty.$$

Now we suppose that $f(I) \cap g(I) \neq \varnothing$ for every nonempty open interval $I \subset \mathbb{R}$. Then the following result holds.

Theorem 2. *If f and g are continuous on $\mathbb{R}$, then $f = g$.*

Proof. We use the same type of reasoning as in the proof of Theorem 1.

Received 19 June 2009; accepted for publication 20 November 2009
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Let $x \in \mathbb{R}$ and n be a natural number, $n \geq 1$. Let $I_n = (x - \frac{1}{n}, x + \frac{1}{n})$. We have $f(I_n) \cap g(I_n) \neq \varnothing$. It results that there exists $y_n \in f(I_n) \cap g(I_n)$. Consequently $y_n = f(u_n) = g(v_n)$ where $u_n \in I_n$ and $v_n \in I_n$. So we have obtained two sequences $(u_n)_n, (v_n)_n \subset \mathbb{R}$ with the properties $u_n \to x$ and $v_n \to x$. From the continuity of f and g at the point x, it results that $y_n = f(u_n) \to f(x)$ and $y_n = g(v_n) \to g(x)$. Consequently $f(x) = g(x)$ and the proof of Theorem 2 is complete. □

Finally, we present an application of Theorem 2.

Problem 1. Let x_0 be a real number. Find all the $\boldsymbol{C}^1$ functions $f\colon \mathbb{R} \to \mathbb{R}$ which satisfy the conditions:

(i) $f(I) \cap f'(I) \neq \varnothing$ for every $I \subset \mathbb{R}$ nonempty open interval;
(ii) $f(0) = x_0$.

To solve the problem, we first use Theorem 2 and we obtain $f = f'$ because f and f' are continuous. Then f is the solution of the Cauchy problem $y' = y$; $y(0) = x_0$, and consequently $f(x) = x_0 \exp(x)$.

References

[1] Deimling, K. (1985). *Nonlinear Functional Analysis.* Springer, Berlin.