Some constraints on the existence of a perfect cuboid

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Abstract

The existence or otherwise of a perfect cuboid is a problem known since at least the time of Euler. This paper uses only the most elementary mathematics to arrive at three previously unpublished constraints on the dimensions of such a cuboid.

A perfect cuboid is a rectangular parallelepiped where all of the seven dimensions — length, breadth, height, the three face diagonals, and the space diagonal — are integers (if all but the space diagonal are integers, it is known as an Euler brick). Thus, if the edges are of lengths a, b, and c, we require solutions to

$$a^2 + b^2 = d_{ab}^2 \tag{1}$$

$$a^2 + c^2 = d_{ac}^2 \tag{2}$$

$$b^2 + c^2 = d_{bc}^2 \tag{3}$$

$$a^2 + b^2 + c^2 = d_s^2, \tag{4}$$

where a, b, c, d_{ab}, d_{ac}, d_{bc}, and d_s are all integers.

It is known that a perfect cuboid, if one exists, has one odd and two even edges (and hence, the space diagonal d_s is odd); that one edge is divisible by 9, and another by 3; that one edge is divisible by 16, and another by 4; that one edge is divisible by 5; and that one edge is divisible by 11 (for example [3], [4], [2], [5]).

We will prove three previously unpublished results regarding primitive perfect cuboids, that is, where the edges have no common factor: (a) that at least one edge must be divisible by 7; (b) that at least one edge must be divisible by 19; and (c) that the prime factors of the space diagonal d_s are all of the form $4n + 1$.

Theorem 1. *One edge of a perfect cuboid must be divisible by 7.*

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Proof. The table of squares mod 7 is

n	$n^2 \pmod 7$
0	0
1	1
2	4
3	2
4	2
5	4
6	1

The squares $\pmod 7$ consist of the set $\{0, 1, 2, 4\}$. We require, for a^2, b^2, $c^2 \pmod 7$, three numbers n_1, n_2, and n_3 from this set (with repetitions allowed) such that n_1+n_2, n_2+n_3, n_3+n_1 and $n_1+n_2+n_3$ are all members of the set. For example, we cannot have edges $a = 1 \pmod 7$ and $b = 2 \pmod 7$, since the squares have sum $5 \pmod 7$, which is impossible. By inspection, one can see that the only possible combinations from this set are

$$\{0,0,0\}, \{0,0,1\}, \{0,0,2\}, \{0,0,4\}, \{0,2,2\}, \{0,4,4\}.$$

All include 0, so at least one square is divisible by 7, and therefore at least one edge is divisible by 7 (since 7 is prime). □

Theorem 2. *One edge of a perfect cuboid must be divisible by 19.*

Proof. The reasoning is similar to that of the previous proof. The squares $\pmod{19}$ consist of the set $\{0, 1, 4, 5, 6, 7, 9, 11, 16, 17\}$. Remarkably, all acceptable sets $\{n_1, n_2, n_3\}$ include 0, therefore some n is divisible by 19, and since 19 is prime, at least one edge is divisible by 19. □

Theorem 3. *All prime factors of the space diagonal d_s are of the form $4n + 1$.*

Proof. The following follow directly from (1), (2), (3), and (4):

$$a^2 + d_{bc}^2 = d_s^2 \tag{5}$$

$$b^2 + d_{ac}^2 = d_s^2 \tag{6}$$

$$c^2 + d_{ab}^2 = d_s^2. \tag{7}$$

Consider a primitive perfect cuboid, that is, one where $\gcd(a, b, c) = 1$. The space diagonal d_s is odd, hence all prime factors of d_s are of the form $4n + 1$ or $4n + 3$. Fermat proved that all prime factors of the hypotenuse of a primitive Pythagorean triple (PPT) are of the form $4n+1$ (see, for example, [1, pp. 288–290]). Therefore, if any of (5), (6), or (7) are PPTs, we are done.

If (5), (6), and (7) are not primitive, then the terms in each of (5), (6), and (7) have a greatest common divisor greater than 1 (of course, the gcd is likely to be different in each case). Thus, if the factors of d_s include a prime, say p, of the form $4n + 3$, then this must divide the greatest common divisor in each case. It follows that a, b, and c must all have the divisor p. This contradicts the assumption that

$\gcd(a, b, c) = 1$. Hence, d_s contains no prime factors of the form $4n + 3$. Thus, since d_s is odd, all its prime factors must be of the form $4n + 1$. $\square$

References

[1] Dantzig, T. (1954). *Number: The Language of Science*. Macmillan, New York.
[2] Guy, R.K. (2004). *Unsolved Problems in Number Theory*. Springer-Verlag.
[3] Kraitchik, M. (1945). On certain rational cuboids. *Scripta Mathematica* **11,** 317–326.
[4] Leech, J. (1977). The rational cuboid revisited. *Amer. Math. Monthly* **84,** 518–533.
[5] Weisstein, E.W. (2009). Perfect Cuboid. From MathWorld — A Wolfram Web Resource. http://mathworld.wolfram.com/PerfectCuboid.html (accessed 22 November 2009).