

Puzzle corner

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Welcome to the Australian Mathematical Society *Gazette's* Puzzle Corner No. 16. From this issue onward, I will be taking over Norman's role of bringing you fun, yet intriguing, puzzles. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites in order to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of \$50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision—that is, my decision—is absolutely final. Please e-mail solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes SA 5095.

The deadline for submission of solutions for Puzzle Corner 16 is 1 May 2010. The solutions to Puzzle Corner 16 will appear in Puzzle Corner 18 in the July 2010 issue of the *Gazette*.

Chocolate blocks

You are given an $m \times n$ block of chocolate which you wish to break into mn unit squares. At each step, you may pick up one piece of chocolate and break it into two pieces along a straight line. What is the minimum number of steps required?



Photo: Zsuzsanna Kilian

Summing products

Consider nonempty subsets of the set $\{1, 2, \dots, N\}$. For each such subset we can compute the product of the reciprocals of each member. Find the sum of all such products.

Point in square

Let $ABCD$ be a square, labelled counter-clockwise in that order. Point P is chosen inside the square, such that $PA = 1$, $PB = 2$ and $PC = 3$. Find the size of angle APB .

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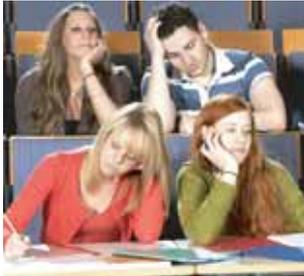


Photo: Bernd Klumpp

Sleepy students

During a particularly long lecture, each of five students fell asleep exactly twice. Furthermore, each pair of students was asleep at the same time, at some point. Prove that there was a moment when at least three students were asleep.

Baffling buckets

- (1) There are three buckets filled with 50, 100 and 150 pebbles, respectively. Each move, you are allowed to take two pebbles out of one bucket, and place one in each of the others. Can you ever have 100 pebbles in each bucket?
- (2) There are 100 distinct buckets, arranged in a circle, each containing some pebbles. Each move, you are allowed to take two pebbles out of one bucket, and place one in each of the neighbouring buckets. Prove that if the initial configuration is achieved again after n moves, then n must be a multiple of 100.
- (3) There are three buckets, each containing some pebbles. Each move, you are allowed to transfer pebbles from one bucket to another if the transfer doubles the number of pebbles in the receiving bucket. Is it always possible to empty one of the buckets?

Solutions to Puzzle Corner 14

The \$50 book voucher for the best submission to Puzzle Corner 14 is awarded to Joe Kupka. Many thanks to Norman Do for collecting and compiling the following solutions.

Matching shoes

Solution by: Xinzi Qiu

We represent each left shoe with a zero and each right shoe with a one. Obviously, we would like to show that if a_1, a_2, \dots, a_{30} is a sequence of 15 zeros and 15 ones, then there must exist 10 consecutive numbers whose sum is five. Write $S_k = a_k + a_{k+1} + \dots + a_{k+9}$ for $1 \leq k \leq 21$ and, in order to obtain a contradiction, let us suppose that $S_k \neq 5$ for all k .

Note that $|S_k - S_{k+1}| \leq 1$ for all k . So if $S_1 > 5$, it follows that $S_2, S_3, \dots, S_{21} > 5$ and this contradicts the fact that $S_1 + S_{11} + S_{21} = 15$. A similar contradiction is obtained if we suppose that $S_1 < 5$, so we must have $S_k = 5$ for some value of k .

Positive subsets

Solution by: Ross Atkins

Let's pair up each subset—including the empty set and X itself—with its complement. The two subsets in each pair cannot both have a positive sum and they cannot both have a nonpositive sum. This is simply because the two sums are integers which add up to 1. Therefore, exactly one subset from each pair has a positive sum, giving 2^{99} subsets in total.

Continued calculation

Solution by: Bruce Bates

Let

$$a = \frac{1}{3 + \frac{1}{4 + \frac{1}{\cdots + \frac{1}{2009}}}}$$

and note that the desired sum is simply

$$\frac{1}{2+a} + \frac{1}{1 + \frac{1}{1+a}} = \frac{1}{2+a} + \frac{1+a}{2+a} = 1.$$

The itinerant queen

Solution by: Joe Kupka

We prove the result for all $m \times n$ chessboards by induction on n . First, we note that the case $n = 1$ is trivial while the case $m = n = 8$ corresponds to the problem we are trying to solve. Now suppose that the problem holds true for all boards with n columns and consider a board with $n + 1$ columns. If we denote the rightmost column by C and the remaining board by B , then one of the following three cases must occur.

- Case 1: In B , both the set of red squares and the set of blue squares are accessible.

In order to obtain a contradiction, suppose that neither the set of blue squares nor the set of red squares in $B \cup C$ is accessible. Denote the squares of C from top to bottom by C_1, C_2, \dots, C_m and the squares immediately to the left of these by L_1, L_2, \dots, L_m . Without loss of generality, suppose that C_1 is blue—then L_1 must be red which in turn implies that C_2 is blue. By continuing this argument, we can show that all of the squares in C are blue, which contradicts the fact that the set of blue squares in $B \cup C$ is not accessible. Hence, the set of blue squares or the set of red squares in $B \cup C$ is accessible.

- Case 2: In B , the set of red squares is accessible, but the set of blue squares is not accessible.

Then each row of B must contain at least one red square, which means that the set of red squares in $B \cup C$ is also accessible.

- Case 3: In B , the set of blue squares is accessible, but the set of red squares is not accessible.

Then each row of B must contain at least one blue square, which means that the set of blue squares in $B \cup C$ is also accessible.

By induction, we conclude that the problem holds true for all $m \times n$ chessboards.

Snail trail

Solution by: Ivan Guo

1. We consider the snail on the Cartesian plane, starting at the origin and traveling one unit up, down, left or right every 15 minutes. Suppose that the snail returns to the origin after taking N steps upward. Since the number of steps upward must equal the number of steps downward, it must make a total of $2N$ vertical steps. Furthermore, since the snail alternates between vertical and horizontal steps, it must make a total of $2N$ horizontal steps. It follows that the snail travels $4N$ units and returns to its starting point after N hours.
2. We will show that if there are $n \geq 10$ people watching, then the snail can crawl at most n metres. Certainly, the snail cannot crawl further than this, since each person watches the snail crawl exactly one metre. To show that the snail can in fact crawl n metres, suppose that all n people watch the snail remain stationary during the first $(n-10)/(n-1)$ hours. After that, the n people take turn to watch the snail crawl one metre in $9/(n-1)$ hours. Then each person has watched the snail for $(n-10)/(n-1) + 9/(n-1) = 1$ hour and the snail has crawled for a total of $(n-10)/(n-1) + 9n/(n-1) = 10$ hours, during which it travels n metres.



Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.