Puzzle corner

Ivan Guo[*]

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 18. Each issue will include a handful of fun, yet intriguing, puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes SA 5095.

The deadline for submission of solutions for Puzzle Corner 18 is 1 September 2010. The solutions to Puzzle Corner 18 will appear in Puzzle Corner 20 in the November 2010 issue of the *Gazette*.

Page numbers

Tom tore out several successive pages from a book. The number of the first page he tore out was 183, and it is known that the number of the last page has the same digits but in another order. How many pages did Tom tear out altogether?

Fraction practice

Franny is practising her fractions on the blackboard. She starts with the following row of 100 fractions:

$$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \cdots \quad \frac{1}{99} \quad \frac{1}{100}.$$

For each of the 99 neighbouring pairs, she calculates the difference and writes the answer in between the pair. Then she erases the original 100 fractions and begins the same subtraction exercise on the 99 new fractions, to obtain 98 new differences. Franny keeps going until she has one fraction left on the board, what is that fraction?

[*] E-mail: ivanguo1986@gmail.com

Invisible point

A point P is marked on a plane with invisible ink. A magical elf can see this ink but no-one else can. You are given a particular square on the plane, and your task is to find out whether P lies inside this square. You are allowed to draw a straight line on the plane and ask the elf on which side of the line the point P lies. The elf will always answer truthfully. If P lies on the line, the elf will say so as well. What is the minimum number of times that you would need to draw a line and ask the elf?

Differing views

An optimist and a pessimist are examining a sequence of numbers. The optimist remarks, 'Oh jolly! The sum of any eight consecutive terms is positive!' But the pessimist interjects, 'Not so fast, the sum of any five consecutive terms is negative.' Can they both be right? How long can this sequence of numbers be?

Coin conundrum

There are coins of various sizes on a table, with some touching others. As often as you wish, you may choose a coin, then turn it over, along with every other coin that it touches. If all coins start out showing heads, is it always possible to change them to all tails using these moves?

Rational points

(1) In the coordinate plane, is it possible to find a 1001-sided polygon with all sides equal and all vertices having rational coordinates?
(2) Is it possible to find 1001 points in the plane with no three being collinear, such that the distance between any pair of points is irrational but the area of any triangle formed by any triple is rational?

Solutions to Puzzle Corner 16

The \$50 book voucher for the best submission to Puzzle Corner 16 is awarded to Joe Kupka. Congratulations!

Chocolate blocks

Solution by: Marston Conder

The minimum number of steps is $mn-1$. In fact, every allowable way of breaking up the chocolate block requires exactly $mn-1$ steps. We start with one piece, end with mn pieces, and each break increases the number of pieces by 1.

Summing products

Solution by: Stephen M^cAteer

Let the required sum be S. We consider the product

$$T = \prod_{i=1}^{N} (1 + \tfrac{1}{i}) = \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{N}\right).$$

Expanding this product, we obtain a sum of terms. Each summand corresponds to some choice of either 1 or $\frac{1}{i}$ from each bracket in the original product. In other words,

$$T = \sum_{P \in \mathcal{P}\{1,2,\ldots,N\}} \prod_{i \in P} \frac{1}{i} = 1 + \sum_{P \in \mathcal{P}\{1,2,\ldots,N\} \setminus \emptyset} \prod_{i \in P} \frac{1}{i} = 1 + S,$$

where $\mathcal{P}\{1, 2, \ldots, N\}$ is the set of all subsets of $\{1, 2, \ldots, N\}$. Finally, we have

$$T = \prod_{i=1}^{N} \left(1 + \frac{1}{i}\right) = \prod_{i=1}^{N} \left(\frac{i+1}{i}\right) = \frac{(N+1)!}{N!} = N + 1$$

and $S = T - 1 = N$.

Sleepy students

Solution by: Gavin Brown

Consider closed intervals $I_1, I_2, \ldots, I_n$ on the number line, ordered by their starting points. In other words

$$\min(I_1) \leq \min(I_2) \leq \cdots \leq \min(I_n).$$

Assume that $I_j \cap I_k \cap I_l = \emptyset$ for all choices of distinct indices j, k and l. Then for each index $k > 1$, the interval I_k can intersect with at most one other interval I_j with $j < k$. Since each intersection of intervals can be uniquely identified by the greatest index involved, and there is no interval before I_1, there can be at most $n - 1$ nonempty intersections of the form $I_j \cap I_k,\ j < k$.

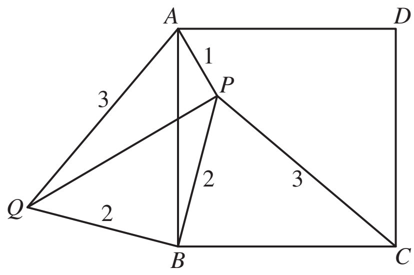
Now let 10 intervals $I_1, I_2, \ldots, I_{10}$ represent the sleep periods of the students, and assume that there is no nonempty triple intersection. By the above arguments there can be at most 9 nonempty double intersections. However, we are told that there are $\binom{5}{2} = 10$ of those, which is a contradiction. So there must be a nonempty triple intersection. Note finally that the intervals come in five pairs, each of which is a disjoint pair. This implies that no pair appears in a nonempty triple intersection, so we really do have three distinct students asleep at the same time.

Point in square

Solution by: Joe Kupka

Rotate $\triangle BPC$ by 90° counter-clockwise about B to form $\triangle BQA$. By construction $\triangle BQP$ is a right isosceles triangle. Hence

$$QP^2 = BQ^2 + BP^2 = 8.$$



Then we have

$$QP^2 + AP^2 = 9 = AQ^2$$

so $\angle APQ = 90°$. Therefore

$$\angle APB = \angle APQ + \angle BPQ = 90° + 45° = 135°.$$

Baffling buckets

Solution by: Sam Krass

1. Consider the set of pebble numbers in modulo 3. In the beginning we have

   $$\{50, 100, 150\} \equiv \{2, 0, 1\} = \{0, 1, 2\}.$$

   This set is actually invariant under the operation, as the possible outcomes, in modulo 3, are

   $$\{0 - 2, 1 + 1, 2 + 1\} = \{1, 2, 0\};$$
   $$\{0 + 1, 1 - 2, 2 + 1\} = \{1, 2, 0\};$$
   $$\{0 + 1, 1 + 1, 2 - 2\} = \{1, 2, 0\}.$$

   Hence the content of the three buckets can never be equal.

2. Label the buckets by $B_1, B_2, \ldots, B_{100}$ in clockwise order, and let the number of pebbles contained by $a_1, a_2, \ldots, a_{100}$ respectively. Define the operation of removing two pebbles from B_i as a *move* by B_i.

   If there is ever a move by the bucket B_i, the affected pebble values are

   $$a_{i-1} \to a_{i-1} + 1; \quad a_i \to a_i - 2; \quad a_{i+1} \to a_{i+1} + 1.$$

   In particular, the value of a_{i+1} has increased. In order to restore the original pebble numbers, there must be a move by B_{i+1} at some point because that is the only move which decreases the value of a_{i+1}.

   Hence if any bucket makes a move, then each of its neighbours also needs to make a move. By repeating the argument, every bucket makes at least one move.

   Now, the net effect of every bucket making a move is zero. Since the order of moves does not affect the outcome (allowing for negative pebble values), we can disregard those 100 moves and are left with the same problem. Therefore the total number of moves must be a multiple of 100.

3. Consider the set S of all possible triples of pebble numbers $b = \{b_1, b_2, b_3\}$, reachable by the doubling move. Since the total number of pebbles is finite, there must exist a triple with a minimal pebble value. In other words, there exists a triple $g = \{g_1, g_2, g_3\} \in S$ with

$$\min(g) \leq \min(b) \quad \text{for all } b \in S.$$

Without loss of generality let $g_1 \leq g_2 \leq g_3$. Assume it is not possible to empty a bucket, so $g_1 > 0$, then we can write $g_2 = kg_1 + r$ where $0 \leq r < g_1$. Consider the binary expansion of k

$$k = \sum_{i=0}^{m} a_i 2^i \quad \text{where } a_i = 0 \text{ or } 1,\ 0 \leq i < m;\ a_m = 1.$$

Now we will double g_1 a total of $m + 1$ times in the following manner. On the ith turn, use g_2 if $a_{i-1} = 1$, otherwise use g_3. The triple at the end of these moves can be computed to be

$$\left\{2^{m+1}g_1,\ g_2 - \sum_{i=0}^{m} a_i 2^i g_1,\ g_3 - \sum_{i=0}^{m}(1 - a_i)2^i g_1\right\}$$
$$= \{2^{m+1}g_1,\ g_2 - kg_1,\ g_3 - (2^{m+1} - 1 - k)g_1\}.$$

To check that this triple is indeed still in S, we just need to check that the terms are nonnegative. Clearly $2^{m+1}g_1 > 0$ and $g_2 - kg_1 = r > 0$. For the remaining term, note that $k \geq 2^m$. So

$$g_3 - (2^{m+1} - 1 - k)g_1 > g_3 - kg_1 \geq g_3 - g_2 \geq 0$$

as required. But now there are $r < g_1$ pebbles in one of the buckets, which contradicts the minimality of g. Therefore we must have $g_1 = 0$, and it is always possible to empty a bucket.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.