A problem involving binomial coefficients

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Abstract

In a particular polynomial, the coefficients change sign three times. We locate these changes of sign with great accuracy.

Introduction

Jim Propp (written communication) was led to consider the following problem.

Suppose the $c_{n,k}$ are defined for $n > 0$, $0 \le k \le 4n$ by the identity

$$(1+2x^3)^{3n} - (2x+x^4)^{3n} = \sum_{k=0}^{4n} c_{n,k} x^{3k}.$$

Then

$$c_{n,k} = 2^k \binom{3n}{k} - 2^{4n-k} \binom{3n}{k-n}.$$

Note that the $c_{n,k}$ are skew–symmetric in the sense that $c_{n,4n-k} = -c_{n,k}$.

The $c_{n,k}$ start off positive. Indeed, it is clear that

$$c_{n,0}, \cdots, c_{n,n-1} > 0$$

(because the second term in $c_{n,k}$ is zero).

Also,

$$c_{n,n} = 2^n \binom{3n}{n} - 2^{3n} > 0$$

provided $n \ge 3$.

However,

$$c_{n,2n} = 0$$

and

$$c_{n,2n-1} < 0,$$

so, if $n \ge 3$, there is a point with $n < k < 2n$ (and by symmetry, another with $2n < k < 3n$) where $c_{n,k}$ changes sign.

The problem now is to find those points where $c_{n,k}$ changes sign.

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The solution

Numerical evidence suggests that $c_{n,k}$ changes sign at points close-ish to the mid-point, $k = 2n$, so I shall investigate what happens when $k = 2n - m$.

Consider $c_{n,2n-m}$; $c_{n,2n-m}$ is roughly 0 when

$$2^{2n-m}\binom{3n}{2n-m} \approx 2^{2n+m}\binom{3n}{n-m},$$

$$2^{2n-m}\frac{(3n)!}{(2n-m)!\,(n+m)!} \approx 2^{2n+m}\frac{(3n)!}{(2n+m)!\,(n-m)!},$$

$$\frac{(2n+m)!\,(n-m)!}{(2n-m)!\,(n+m)!} \approx 2^{2m},$$

$$\frac{(2n+m)(2n+m-1)\cdots(2n-m+1)}{(n+m)(n+m-1)\cdots(n-m+1)} \approx 2^{2m},$$

$$\frac{(2n+m)(2n+m-1)\cdots(2n-m+1)}{(2n+2m)(2n+2m-2)\cdots(2n-2m+2)} \approx 1,$$

$$\frac{(4n^2-1^2)(4n^2-2^2)\cdots(4n^2-(m-1)^2)}{(4n^2-2^2)(4n^2-4^2)\cdots(4n^2-(2m-2)^2)} \approx \frac{2n+2m}{2n+m},$$

$$\frac{\left(1-\frac{1^2}{4n^2}\right)\left(1-\frac{2^2}{4n^2}\right)\cdots\left(1-\frac{(m-1)^2}{4n^2}\right)}{\left(1-\frac{2^2}{4n^2}\right)\left(1-\frac{4^2}{4n^2}\right)\cdots\left(1-\frac{(2m-2)^2}{4n^2}\right)} \approx \frac{1+\frac{m}{n}}{1+\frac{m}{2n}}.$$

If we take logs, we find

$$\frac{-(1^2+2^2+\cdots+(m-1)^2)+(2^2+4^2+\cdots+(2m-2)^2)}{4n^2}$$
$$+\frac{-(1^4+2^4+\cdots+(m-1)^4)+(2^4+4^4+\cdots+(2m-2)^4)}{32n^4}+\cdots$$
$$\approx \frac{m}{n}-\frac{m^2}{2n^2}+\frac{m^3}{3n^3}-+\cdots\;-\frac{m}{2n}+\frac{m^2}{8n^2}-\frac{m^3}{24n^3}+-\cdots,$$

or,

$$\frac{m^3-\frac{3}{2}m^2+\frac{1}{2}m}{4n^2}+\frac{3m^5-\frac{15}{2}m^4+5m^3-\frac{1}{2}m}{32n^4}+\cdots \approx \frac{m}{2n}-\frac{3m^2}{8n^2}+\frac{7m^3}{24n^3}-\cdots.$$

If we multiply by $\frac{4n^2}{m}$, this becomes

$$m^2+\frac{1}{2}+\frac{3m^4-\frac{15}{2}m^3+5m^2-\frac{1}{2}}{8n^2}+\cdots \approx 2n+\frac{7m^2}{6n}+\cdots.$$

A first approximation is

$$m \approx \sqrt{2n}.$$

If we then seek a better approximation, of the form $m \approx \sqrt{2n}(1+x)$, we find

$$2n(1+2x)+\frac{1}{2}+\frac{12n^2}{8n^2} \approx 2n+\frac{14n}{6n},$$

or,

$$4nx + \frac{1}{2} + \frac{12}{8} \approx \frac{14}{6},$$

from which

$$x \approx \frac{1}{12n}$$

and

$$m \approx \sqrt{2n}\left(1 + \frac{1}{12n}\right).$$

So changes of sign in $c_{n,k}$ occur approximately at $k \approx 2n \pm \sqrt{2n}(1 + \frac{1}{12n})$ (as well as at $k = 2n$).

For $n = 200$, this says that $c_{200,k}$ changes sign when $k \approx 379.99$ (and at $k = 400$, and at $k = 420.01$). In fact, $c_{200,379} \approx 6.6 \times 10^{281}$, while $c_{200,380} \approx -6.1 \times 10^{279}$. Linear interpolation between these gives almost perfect agreement.