Zeta-functions through the 2-adic looking glass

Daniel Delbourgo*

Introduction

Your typical undergraduate maths student first encounters Riemann's zeta-function via its Dirichlet series expansion $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which converges only within the right half-plane $\mathrm{Re}(s) > 1$. Let's play devil's advocate, and ask the ridiculous

Question. Does $\zeta(-k)$ also coincide with $1^k + 2^k + 3^k + \dots$ at positive integers k?

At this point your undergraduate student will give you a withering look, and explain that the terms x^k diverge alarmingly as x grows, and so you are wasting their time.

The purpose of this note is to explain that whilst such a formula is impossible, it also happens to be true! Imagine the unlikely scenario where the Australian Democrats join forces with Family First, and are swept to power in a total landslide. Their first parliamentary bill is to abolish the usual notion of distance, and replace it with the rule that the distance between two integers x and y is 2^{-N} whenever $x - y$ is exactly divisible by 2^N.

Completing $\mathbb{Z}$ with respect to this new distance, one obtains the 2-adic integers

$$\mathbb{Z}_2 := \left\{ a_0 + a_1 \times 2 + a_2 \times 2^2 + a_3 \times 2^3 + \cdots \text{where each } a_i \text{ is either 0 or 1} \right\}.$$

Under this bizarre 2-adic topology, we'll proceed to show (for positive integers k) there exists an equally bizarre formula

$$\begin{aligned}\zeta(1-2k) = \frac{1}{9^k - 1} &\times \left(1 + 2^{2k-1} + 4^{2k-1} + 8^{2k-1} + 16^{2k-1} + \cdots\right) \\ &\times \big(3^{2k-1} - 5^{2k-1} + 9^{2k-1} - 11^{2k-1} \\ &\qquad + 15^{2k-1} - 17^{2k-1} + 21^{2k-1} - 23^{2k-1} + \cdots\big).\end{aligned}$$

The first bracket converges unconditionally since $2^{n(2k-1)} \longrightarrow 0$ rapidly as $n \to \infty$. To make the second bracket converge is a bit trickier, as each of the terms in the series we're summing has 2-adic norm equal to 1, so they aren't getting smaller.

Received 14 October 2008; accepted for publication 26 June 2009.
*School of Mathematical Sciences, Monash University, VIC 3800.
E-mail: daniel.delbourgo@sci.monash.edu.au

As we shall soon see, if one groups together (ignoring their sign) all terms $\leq 2^{2N+1}$ and then takes the limit as $N \to \infty$, the partial sums converge rather quickly[1].

This method is based on the following congruence modulo 2^{2n+1}, satisfied by all Bernoulli numbers. Recall these numbers occur as coefficients in the Taylor series

$$\frac{Z}{\exp(Z) - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \times Z^k.$$

For instance, when the index $k \geq 3$ is *odd* then the corresponding B_k will be 0. The result below is essentially a converse statement to some older works in [1] and [4].

Proposition 1. *If both $k, n \geq 1$ are integers, then*

$$\frac{(3^{2k}-1)(2^{2k-1}-1)}{2k} B_{2k} \equiv \sum_{\substack{m=1 \\ m \equiv 3,9 (\mathrm{mod}\, 12)}}^{2\times 4^n} m^{2k-1} - \sum_{\substack{m=1 \\ m \equiv 5,11 (\mathrm{mod}\, 12)}}^{2\times 4^n} m^{2k-1} \pmod{2 \times 4^n}.$$

A natural way to understand these formulae is to deform them over the ring $\mathbb{Z}_2$. The 2-adic L-function $\boldsymbol{L}_{\text{2-adic}}(s)$ constructed by Kubota-Leopoldt has a simple pole at the point $s = 1$, and at negative odd integers interpolates the zeta-values

$$\boldsymbol{L}_{\text{2-adic}}(1-2k) = \left(1 - 2^{2k-1}\right) \times \zeta\left(1-2k\right).$$

At negative even integers $\boldsymbol{L}_{\text{2-adic}}(-2k)$ coincides with the twisted value $\zeta(-2k, \chi)$, where χ denotes the quadratic character of the Gaussian integers (see [5, p. 57]).

Theorem 1. *For all $s \in \mathbb{Z}_2 - \{1\}$, we have the equality of rigid-analytic functions*

$$\boldsymbol{L}_{\text{2-adic}}(s) = \frac{1}{< 3 >^{1-s} - 1} \times \lim_{n\to\infty} \left(\sum_{\substack{m=1 \\ m \equiv 9,11 (\mathrm{mod}\, 12)}}^{2\times 4^n} < m >^{-s} - \sum_{\substack{m=1 \\ m \equiv 3,5 (\mathrm{mod}\, 12)}}^{2\times 4^n} < m >^{-s} \right),$$

where $< m >^{-s} := \exp_2(-s \log_2(m))$ is the continuous 2-adic analogue of m^{-k}.

In particular, this result reveals zeta's hidden analytic shape (as a Dirichlet series) at that most obnoxious of customers, namely the prime $p = 2$.

The proofs

Let Λ be the power series ring $\mathbb{Z}_2[[X]]$, so Λ is isomorphic to the projective limit $\varprojlim_N \mathbb{Z}_2[X]/((1+X)^{2^N} - 1)$. One writes μ_{2^N} for the group of 2^Nth roots of unity.

[1] We can even go a stage further and rewrite this expansion in terms of Euler products, however one must replace the 2-adic topology with a finer 'shadow topology', as is explained in the article [3].

Given an element $F \in \Lambda$, then

$$F(X) \equiv R_N(X) \qquad \big(\mathrm{mod}\,\big((1+X)^{2^N}-1\big)\cdot \mathbb{Z}_2[[X]]\big)$$

where $R_N(X) = 2^{-N}\sum_{\alpha\in\mu_{2^N}} F(\alpha^{-1}-1)\sum_{m=0}^{2^N-1}\alpha^m(1+X)^m$ is a polynomial with degree $< 2^N$ — for an easy proof of this formula, see [2, Lemma 2.1].

We introduce the logarithmic differential operator $D = (1+X)(\frac{\mathrm{d}}{\mathrm{d}X})$, and consider Coleman's projector $\psi \in \mathrm{End}_{\mathbb{Z}_2}(\Lambda)$ defined by $\psi\circ F(X) := \frac{1}{2}F(X) - \frac{1}{2}F(-X-2)$. For any pair of integers $k, N \geq 1$,

$$D^{2k-1}\psi\circ F(X) \equiv D^{2k-1}\psi\circ R_N(X) \qquad (\mathrm{mod}\,(2,X)^N),$$

since $D^{2k-1}\circ\psi(((1+X)^{2^N}-1)\cdot\mathbb{Z}_2[[X]]) \subset ((1+X)^{2^N}-1)\cdot\Lambda + 2^N\Lambda \subset (2,X)^N$.

Definition 1. From now onwards, we'll set $F(X) := X^{-1} - 3((1+X)^3-1)^{-1}$.

By the previous discussion, if $N = 2n+1$ then

$$D^{2k-1}\psi\circ F(0) \equiv D^{2k-1}\psi\circ R_{2n+1}(0) \qquad (\mathrm{mod}\,2^{2n+1}).$$

To compute the left-hand side, observe that

$$\begin{aligned}
D^{2k-1}\psi\circ F(0) &= (1-2^{2k-1})D^{2k-1}F(0)\\
&= (1-2^{2k-1})\frac{\mathrm{d}^{2k-1}F(\exp(Z)-1)}{\mathrm{d}Z^{2k-1}}\bigg|_{Z=0}\\
&= (1-2^{2k-1})\frac{\mathrm{d}^{2k-1}}{\mathrm{d}Z^{2k-1}}\frac{1}{Z}\left(\frac{Z}{\exp(Z)-1} - \frac{3Z}{\exp(3Z)-1}\right)\bigg|_{Z=0}\\
&= (1-2^{2k-1})\frac{\mathrm{d}^{2k-1}}{\mathrm{d}Z^{2k-1}}\frac{1}{Z}\left(\sum_{n=0}^{\infty}B_n\frac{Z^n}{n!} - \sum_{n=0}^{\infty}B_n\frac{(3Z)^n}{n!}\right)\bigg|_{Z=0}\\
&= (1-2^{2k-1})(1-3^{2k})\frac{B_{2k}}{2k}.
\end{aligned}$$

Computing the right-hand side is more challenging; the proof of the following lemma will be deferred to the final section.

Lemma 1. *For $F(X)$ and $R_N(X)$ as above, we have the equality*

$$D^{2k-1}\psi\circ R_{2n+1}(0) = \sum_{\substack{m=1\\ m\equiv 3,9 (\mathrm{mod}\,12)}}^{2\times 4^n} m^{2k-1} - \sum_{\substack{m=1\\ m\equiv 5,11 (\mathrm{mod}\,12)}}^{2\times 4^n} m^{2k-1}.$$

Combining the left- and right-hand sides together, yields the 2-adic congruence[2]

$$(1-3^{2k})(1-2^{2k-1})\frac{B_{2k}}{2k} \equiv \sum_{\substack{m=1 \\ m\equiv 3,9 (\mathrm{mod}\, 12)}}^{2\times 4^n} m^{2k-1} - \sum_{\substack{m=1 \\ m\equiv 5,11 (\mathrm{mod}\, 12)}}^{2\times 4^n} m^{2k-1} \quad (\mathrm{mod}\, 2^{2n+1})$$

which is identical to the statement of Proposition 1.

To deduce Theorem 1, we simply observe that $(1-3^{2k}) \times (1-2^{2k-1})(B_{2k}/2k)$ is uniquely interpolated by the Iwasawa function $(1- < 3 >^{1-s}) \times -\boldsymbol{L}_{2\text{-adic}}(s)$ at negative integers $s = 1-2k$.

On the other hand $< m >^{-s} |_{s=1-2k}$ equals m^{2k-1} if $m \equiv 5, 9 (\mathrm{mod}\, 12)$, and equals $-m^{2k-1}$ if $m \equiv 3, 11 (\mathrm{mod}\, 12)$. When we specialise at odd negative integers, $(1- < 3 >^{1-s}) \times -\boldsymbol{L}_{2\text{-adic}}(s)|_{s=1-2k}$ must therefore be congruent modulo 2^{2n+1}, to

$$- \sum_{\substack{m=1 \\ m\equiv 3 (\mathrm{mod}\, 12)}}^{2\times 4^n} < m >^{-s} + \sum_{\substack{m=1 \\ m\equiv 9 (\mathrm{mod}\, 12)}}^{2\times 4^n} < m >^{-s}$$
$$- \sum_{\substack{m=1 \\ m\equiv 5 (\mathrm{mod}\, 12)}}^{2\times 4^n} < m >^{-s} + \sum_{\substack{m=1 \\ m\equiv 11 (\mathrm{mod}\, 12)}}^{2\times 4^n} < m >^{-s}$$

evaluated at $s = 1-2k$. Taking the limit as $n \to \infty$ and dividing by the Euler factor $(< 3 >^{1-s} -1)$, the demonstration of the theorem is complete.

The moments of $\psi \circ R_{2n+1}(X)$

It remains to supply the missing proof of Lemma 1.

We first remark that the idempotent ψ kills off terms of the form $(1+X)^m$ with m even, and preserves them for odd m. It follows from our formula for $R_N(X)$ that

$$\psi \circ R_{2n+1}(X) = 2^{-(2n+1)} \sum_{\alpha\in\mu_{2^{2n+1}}} F(\alpha^{-1}-1) \sum_{\substack{m=0 \\ m \text{ odd}}}^{2^{2n+1}-1} \alpha^m (1+X)^m.$$

If we define $\Omega_n^{(m)}(F) := 2^{-(2n+1)} \sum_{\alpha\in\mu_{2^{2n+1}}} F(\alpha^{-1}-1)\alpha^m$, clearly

$$D^{2k-1}\psi \circ R_{2n+1}(0) = \sum_{\substack{m=1 \\ m \text{ odd}}}^{2^{2n+1}} \Omega_n^{(m)}(F) \times m^{2k-1}$$

for all values of the positive integers n and k.

Thus Lemma 1 above will certainly follow, provided one can establish the following.

[2]There is an alternate way to obtain this congruence using Iwasawa's 'Stickelberger ideals' method, specifically by twisting at $c=3$; however our methods are more low-brow (albeit slightly lengthier).

Key Claim. For all odd integers m in the range $1 \leq m \leq 2^{2n+1}$,

$$\Omega_n^{(m)}(F) = \begin{cases} +1 & \text{if } m \equiv 0 (\bmod 3) \\ 0 & \text{if } m \equiv 1 (\bmod 3) \\ -1 & \text{if } m \equiv 2 (\bmod 3) \ . \end{cases}$$

To verify this assertion, we plug the definition of $F(X)$ into the Ωs, so that

$$\begin{aligned} \Omega_n^{(m)}(F) &= 2^{-(2n+1)} \Bigg(F(0) + \sum_{\substack{\alpha \in \mu_{2^{2n+1}} \\ \alpha \neq 1}} \frac{\alpha^m}{\alpha^{-1} - 1} - 3 \sum_{\substack{\alpha \in \mu_{2^{2n+1}} \\ \alpha \neq 1}} \frac{\alpha^m}{\alpha^{-3} - 1} \Bigg) \\ &= 2^{-(2n+1)} \Bigg(1 + \sum_{\substack{\zeta \in \mu_{2^{2n+1}} \\ \zeta \neq 1}} \frac{\zeta^{-m}}{\zeta - 1} - 3 \sum_{\substack{\zeta \in \mu_{2^{2n+1}} \\ \zeta \neq 1}} \frac{\zeta^{-m/3}}{\zeta - 1} \Bigg). \end{aligned}$$

We now appeal to a technical result.

Lemma 2. *For any $x \in \mathbb{Z}_2$, let $\lambda_n(x)$ denote the element of $\{1, 2, 3, \ldots, 2^{2n+1}\}$ congruent to x modulo $2^{2n+1}\mathbb{Z}_2$. Then*

$$\sum_{\substack{\zeta \in \mu_{2^{2n+1}} \\ \zeta \neq 1}} \frac{\zeta^x}{\zeta - 1} = \tfrac{1}{2} + 4^n - \lambda_n(x).$$

Proof. Without loss of generality, we assume that x is actually a positive integer. To simplify matters let's work inside the complex numbers. Firstly, if $x = 1$ then

$$\begin{aligned} \sum_{\substack{\zeta \in \mu_{2^{2n+1}} \\ \zeta \neq 1}} \frac{\zeta}{\zeta - 1} &= \sum_{\zeta \neq 1} \frac{\zeta - 1}{\zeta - 1} + \sum_{\zeta \neq 1} \frac{1}{\zeta - 1} \\ &= 2^{2n+1} - 1 + \sum_{\zeta \neq 1} \frac{\overline{\zeta} - 1}{(\zeta - 1)(\overline{\zeta} - 1)} \\ &= 2^{2n+1} - \frac{3}{2} - \sum_{\zeta \neq \pm 1} \frac{\mathrm{Re}(\overline{\zeta}) + \mathrm{Im}(\overline{\zeta}) - 1}{2\mathrm{Re}(\overline{\zeta}) - 2} \\ &= 2^{2n+1} - \frac{3}{2} - \left(\frac{2^{2n+1} - 2}{2} \right) - \sum_{\zeta \neq \pm 1} \frac{\mathrm{Im}(\overline{\zeta})}{2\mathrm{Re}(\overline{\zeta}) - 2}. \end{aligned}$$

The right-most sum contributes nothing, since $\mathrm{Im}(\overline{\zeta})$ cancels with $\mathrm{Im}(\zeta)$. It follows that

$$\sum_{\substack{\zeta \in \mu_{2^{2n+1}} \\ \zeta \neq 1}} \frac{\zeta^1}{\zeta - 1} = 2^{2n+1} - \frac{3}{2} - (2^{2n} - 1) = \frac{1}{2} + 4^n - \lambda_n(1)$$

as predicted.

On the other hand, if $x > 1$ then

$$\sum_{\substack{\zeta\in\mu_{2^{2n+1}}\\ \zeta\neq 1}} \frac{\zeta^x}{\zeta-1} - \sum_{\substack{\zeta\in\mu_{2^{2n+1}}\\ \zeta\neq 1}} \frac{\zeta^{x-1}}{\zeta-1} = \sum_{\zeta\neq 1}\zeta^{x-1} = -1$$

as $x \not\equiv 1 (\bmod 2^{2n+1})$. The result follows by this simple induction.

Applying the above lemma twice to the moments Ω, we discover that

$$\begin{aligned}\Omega_n^{(m)}(F) &= 2^{-(2n+1)}\left(1 + \left(\frac{1}{2} + 4^n - \lambda_n(-m)\right) - 3\left(\frac{1}{2} + 4^n - \lambda_n\left(-\frac{m}{3}\right)\right)\right)\\ &= -1 + \frac{3\lambda_n(-m/3) - \lambda_n(-m)}{2^{2n+1}}\\ &= -1 + \frac{3\lambda_n(-m/3) - (2\times 4^n - m)}{2^{2n+1}}.\end{aligned}$$

Let's subdivide the calculation into three distinct pieces.

Case (I): $m \equiv 0 (\bmod 3)$. This one is easy; here $\lambda_n(-m/3) = 2\times 4^n - m/3$, so

$$\Omega_n^{(m)}(F) = -1 + \frac{3(2\times 4^n - m/3) - (2\times 4^n - m)}{2^{2n+1}} = +1.$$

In the remaining cases, we profit from the following stroke of luck.

Trivial fact. The positive integer $(1 + 2\times 4^n)/3$ is congruent to 3^{-1} modulo 2^{2n+1}.

Case (II): $m \equiv 1 (\bmod 3)$. Under the above condition,

$$-\frac{m}{3} = -\frac{1}{3} - \left(\frac{m-1}{3}\right) \equiv -\left(\frac{1+2\times 4^n}{3}\right) - \left(\frac{m-1}{3}\right) \text{modulo } 2\times 4^n,$$

hence

$$\lambda_n\left(-\frac{m}{3}\right) = 2\times 4^n - \left(\frac{1+2\times 4^n}{3}\right) - \left(\frac{m-1}{3}\right) = \frac{4^{n+1}-m}{3}.$$

It follows that

$$\Omega_n^{(m)}(F) = -1 + \frac{3((4^{n+1}-m)/3) - (2\times 4^n - m)}{2^{2n+1}} = 0.$$

Case (III): $m \equiv 2 (\bmod 3)$. Finally, in this situation

$$-\frac{m}{3} = \frac{1}{3} - \left(\frac{m+1}{3}\right) \equiv \left(\frac{1+2\times 4^n}{3}\right) - \left(\frac{m+1}{3}\right) \text{modulo } 2\times 4^n,$$

which means $\lambda_n(-m/3) = (2\times 4^n - m)/3$. Back substituting into Ω, we conclude that

$$\Omega_n^{(m)}(F) = -1 + \frac{3((2\times 4^n - m)/3) - (2\times 4^n - m)}{2^{2n+1}} = -1.$$

The proof is finished.

References

[1] Boyd, D. (1994). A p-adic study of the partial sums of the harmonic series. *Experiment. Math.* **3,** 287–302.

[2] Delbourgo, D. (2006). A Dirichlet series expansion for the p-adic zeta function. *J. Aust. Math. Soc.* **81,** 215–224.

[3] Delbourgo, D. (2009). The convergence of Euler products over p-adic number fields. *Proc. Edinburgh Math. Soc.* (to appear).

[4] Washington, L.C. (1998). p-adic L-functions and sums of powers. *J. Number Theory* **69,** 50–61.

[5] Washington, L.C. (1997). *Introduction to Cyclotomic Fields* (Graduate Texts in Mathematics **83**). Springer-Verlag, New York.