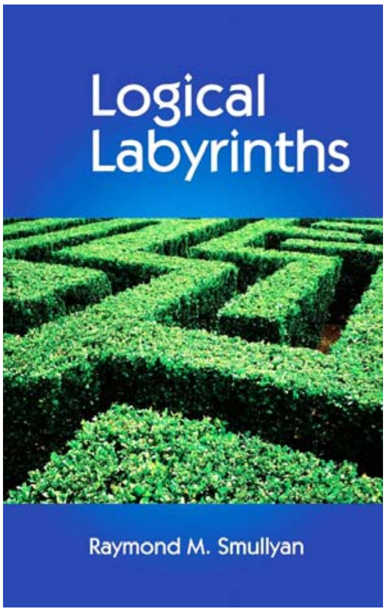
Book reviews

Logical Labyrinths

Raymond Smullyan
A.K. Peters, 2009, ISBN: 978-56881-443-8

A classic example in Greek philosophy is the Liar Paradox. Epimenides, the Cretan, said 'All Cretans are liars'. If we impose a requirement of consistency, such as that the speaker either lies or tells the truth, the statement is no longer a paradox, but a fallacy, because neither a liar nor a truth-teller could consistently state 'I am a liar'. In spite of appearances, this example is not a triviality. In fact, by changing 'true' to 'provable', the proposition becomes 'This sentence is unprovable'. Thus there is a direct line from the liar paradox to Gödel's Undecidability Theorem, which states that every consistent theory which is expressive enough to admit a proposition asserting its own provability must contain a statement which is neither provable nor unprovable.

One purpose of Smullyan's new book is to trace this development. He begins by transforming the liar paradox into an amusing logical riddle: in a consistent world, C states that B is lying when he claims that A stated that he (A) is a liar. Is C a liar or a truth-teller? If you have followed what I have said so far, you will see immediately that C must be a truth-teller.



Examples such as this introduce the second purpose of Smullyan's book, a layman's introduction to classical propositional and first-order (quantificational) logic. He begins by developing informal logic, based on the notion of a population of Knights, that is, consistent truth-tellers, and Knaves or consistent liars. Using these ideas, he introduces in clear elementary terms the notions of logical connectives, rules of inference and systematic ways to establish the truth or falsity of compound propositions. This section of the book is both rigorous and easy to read, containing many jokes, anecdotes and quotations from unexpected sources.

If this looks familiar, you have probably read some of Smullyan's earlier best-selling popularisations of logic, such as *What is the Name of this Book?* (1978), *The Lady or the Tiger* (1982) and *To Mock a Mocking Bird and other Logic Puzzles* (1985). However, the book under review contains many more intriguing puzzles than these earlier volumes, including for example Knights and Knaves who always tell the truth or lie about their beliefs and their wives who do the opposite, but some of whom are mad; that is, their beliefs are consistently incorrect!

In fact this book goes much farther. The meat of the volume is the formalisation into symbolic logic of the earlier intuitive concepts. This is the third purpose of Smullyan's book: a textbook for a university course in mathematical logic. Reprising the earlier ideas, he introduces the languages of propositional and first order logic and various equivalent sets of connectives, axioms and rules of inference. A noteworthy feature in his development of formal proofs in propositional logic is 'proof by tableaux'. This procedure, largely due to the author himself, has three major advantages over truth tables:

(1) It is more efficient, systematically advancing from the premises to the conclusion.
(2) When a proposition is false, the method not only demonstrates this, but shows precisely why it is false.
(3) The method can be extended to first order logic.

Before proving the fundamental theorems concerning compactness, correctness and completeness of first order logic, the author interpolates more elementary chapters on infinity and on mathematical induction, both on ordinals and on well-founded trees. Incidentally, for an intuitive introduction to induction the author does not employ the badly worn cliché of falling dominoes, but rather the metaphor of a note to the milkman stating:

(1) leave milk today and
(2) if you leave milk any day, leave it the next day too.

This chapter on induction, containing several original applications to geometry and algebra, is a suitable introduction to mathematical induction for any student, not just one learning logic.

The author now explains and proves in considerable detail some of the main theorems of classical first order logic, including the Skolem–Löwenheim theorem, Craig's interpolation lemma and its consequences, Robinson's consistency theorem and Beth's definability theorem.

As a *tour de force* before the final chapter on incompleteness and Gödel's Theorems, Smullyan unifies most of these classical results by showing that *all* the major results of classical first order logic are consequences of a single theorem due to the author, which he calls the 'Abstract Model Existence Theorem'.

Finally, Smullyan provides an extensive discussion of the Incompleteness Phenomenon. He shows that Gödel's original theorem, that under the Peano axioms, the theory of ordinary whole numbers contains undecidable statements, can be extended to show that any sufficiently expressive consistent theory contains a sentence that is true but not provable.

There is much to like in this volume. Firstly, Smullyan is a master of mathematical prose. As a result, nearly every page, even those containing dense mathematical argument, is a pleasure to read.

Secondly, the book contains within the text around 200 problems of varied levels of difficulty and every one of them is given a complete solution at the end of the chapter.

Thirdly, while holding no brief for intuitionism, Smullyan expounds it clearly and describes its advantages. He explains the difference between classical and intuitionistic axioms and rules of inference, and points out precisely which steps in certain proofs are not intuitionistically acceptable.

So far, this review has been wholly positive. Unfortunately, I must also point out that the book is plagued by proof-reading faults. Most are harmless, if annoying to a pedantic reader. But in one egregious example, a whole line in the exposition is omitted.

To finish on a brighter note, here is another ingenious riddle from the island of Kights and Knaves. One day, a man was tried for a crime. First the prosecutor claimed: 'If he is guilty, then he had an accomplice'. The defense attorney retorted: 'That's not true!'

The judge didn't know whether either prosecutor or defense attorney was a Knight or a Knave, so was unable to decide whether to convict or acquit. Later, however, he did find out the type of the defense attorney, and was then able to come to a decision. Did he convict or acquit?

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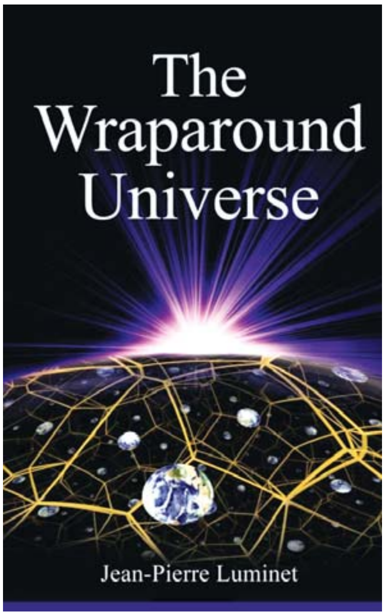
The Wraparound Universe

Jean-Pierre Luminet

A.K. Peters, 2008, ISBN: 978-56881-309-7

The first things that will strike the reader about this book are the multiplicity of chapters (there are 45) and consequently the brevity of some of those chapters. The shortest chapter, Chapter 3, is just two pages long. One of those pages has over half of it devoted to a figure and the lead page of the chapter has around a third devoted to the title and a quote. So while there is not a great deal of reading in that chapter, others are very different. The other nice touch was a little right or left arrow with a number directly above it associated with a paragraph directing the reader to a page (the number above the arrow) where a fuller treatment or similar idea was detailed: a great idea and the closest form in the written media for a hypertext link.

Let me say at the outset that this was a great and interesting read. The title was catchy and the interior lived up to the title. The multiple chapters, with their short content provided a fast pace (or can I say racey) style that was never boring but always instructive. I would recommend the book and believe it is suitable for undergraduate physics or mathematics students.



The book is divided into two parts: the first, Chapters 1 to 23, deals with the shape of the Universe, while the second part covers folds in the universe. In the first part Luminet starts with some definitions of a 'mathematical space', a 'physical space', 'space-time' and lastly 'the Universe'. From there he moves quickly through the size of space to the topology of space, while considering the different levels of geometry on which that space is defined. There are some simple figures which describe these types of geometry along with good written descriptions of the topology he is trying to elucidate. There is little mathematical description or rigour, so one will have to look elsewhere for that, but his analogies in some cases are very clever and clear. One example of this is when he describes space-time foam by flying over water in an aircraft at different altitudes, which is something most people can identify or envisage. Of course, in a work such as this, one cannot help but mention Einstein's relativity and the various big bang models. Those covered are the closed, open, Friedmann–Lemaître and Einstein–de Sitter models of the big bang, and his explanations here are concise and clear. Obviously related to all of this is the curvature of the universe. He covers the classical Euclidean geometry to the Riemannian and Lobachevsky geometries and as to whether their topologies are finite or infinite as well as whether they are open or closed. All the time Luminet provides analogies or figures to explain the geometric concepts he is describing. Many a time a nod is given to the work of mathematicians years before there was any practical result of their work.

> Nevertheless, throughout the twentieth century mathematicians have discovered fascinating forms, known practically to themselves alone, which can be used by physicists for the description of the real Universe.

He does add extra information about the topic at hand or those ideas coming up, devoting an entire chapter to fascinating shapes. Here the well-known Möbius strip and Klein bottle are covered as well as the less well known, except perhaps for topologists, like Seifert–Weber hyperbolic space and Poincare's spherical space. As well, mention is made of Jeffrey Weeks discovery in 1985 of the smallest hyperbolic shape, so one can see that there is a good deal of interesting material for the budding mathematician, particularly if they have a topological bent. Nor is the physicist left out, with gravitational lensing (the chapter is titled 'Hunting for Ghosts'), quasars and other interesting items from the galactic zoo.

The second part of the book moves away somewhat from the mathematical concepts to the cosmological with the opening chapter covering a brief history of space and how, down though the ages, the scientists have alternated between space being infinite or finite. Luminet, of course, proposes that the universe may be finite depending on its topology and asks what the future will hold based on these ideas.

He also asks about the age-old questions of cosmology. The first, geometric cosmology, which speculates on its shape and frontiers and the local and global properties of space-time. The second, physical cosmology, which looks at the material processes taking place in the universe or at some previous stage of its evolution. And lastly, observational cosmology, which deals with the facts and nothing but the facts — background radiation, the distribution of quasars and galaxy clusters, etc. From here he dips his toe into the four-dimensional space-time metric tensor, Einstein's cosmological constant λ, homogeneous Friedmann–Lemaître models of the universe and some of their variants. But in all these cases the operative word is 'dips' so there is not a great deal of depth. Though that said, it is good to see that he tackled them. Incidentally this is one of the longer chapters of the book.

No book of this scope would be complete without mention of dark matter and the challenge it presents in finding it as well as exactly how much exists in galaxies and galaxy clusters. This leads well into cosmic background radiation, which has a nice historical introduction along with the work of the COBE satellite. As he comes to his conclusion he covers the interesting areas of symmetry and the classical five regular polyhedra as well as listing some mathematical tools in order to try and classify our three-dimensional space. Does it have zero curvature, or positive or perhaps even negative curvature. Either way, Luminet illustrates his ideas and concepts well with many diagrams and figures. The colour plates in the centre of the book are an added bonus. So do your mind a favour, get a copy to wrap it around these ideas that in some cases just aren't intuitive. Recommended!

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Emmy Noether: The Mother of Modern Algebra

Margaret Tent
A.K. Peters, 2008, ISBN-13: 978-1568814308

Emmy Noether: The Mother of Modern Algebra by Margaret Tent begins: 'This is the life story of Emmy Noether'. Unfortunately this statement gives the unsuspecting reader a belief that the book is a factual account, but the writer then says:

> ... Because no one expected Emmy to grow up to be an important scientist, the records of her early life are sketchy. After all, it was assumed that she would grow up to be a wife and mother .... However, we should forgive her parents for not foreseeing her remarkable future given the limited options available to women at the time. (p. ix)

The only problem with this is that Emmy had three younger brothers; one, Fritz, who also grew up to be a mathematician, but his life is even less documented, so we can only conclude Max and Ida Noether had no aspirations for their sons either.

The writer continues: 'Since we know none of the details ... my only option was to construct plausible scenes of her childhood in a way that I think they might have happened ...' (p. x). The fiction is not exclusive to Noether's childhood and dominates the text mainly in the form of dialogue. Consequently, this book is more like a novel based on the life of Emmy Noether rather than being a biography. The book is written for a young audience. However, there is a very real danger of myth-making when this work has the credentials of a biography and therefore a stamp of authority. The writer often embeds historical facts and speech from real people within her fictitious dialogue to a point that there is little distinction between the writer's creation and true facts; for example, when David Hilbert defends Emmy Noether by reminding his colleagues they are a university not a bathing establishment (p. 80) the wording used is inconsistent with other quotes of the same situation [1] suggesting Tent simply reworded the retort of Hilbert for her own convenience. There is no way to gauge how often this occurs in the book.

Verging on a hagiography, Margaret Tent depicts Emmy Noether as an 'average girl'; clever, but not promising enough to have her parents' support, growing up in a 'normal' house and groomed into domestic bliss of piano lessons and embroidery. In the process, we are given a picture of a not-so-attractive female, educated mainly in languages and basic studies, suddenly struck by a lightning bolt at the age of 18 years to bloom into a mathematical super-genius.

While it is impossible to determine the source of intelligence — either nature or nurture — it is disturbing in the book to find the author's depiction of Emmy's parents as weak characters with whom Emmy Noether gains neither moral nor academic support. Particularly, Max Noether is described as a mathematician who doesn't see mathematics as the most important thing in the world; he doesn't bring his work home, and doesn't have peers around for meals except for one occasion when Emmy Noether meets her future PhD mentor, Professor Paul Gordon. Professor Noether not only keeps maths out of the house until his boys are old enough to study algebra but he refuses to coach Emmy and initially supports the mother to prevent his daughter ever knowing what mathematics is. In fictional conversations, Emmy doesn't know what her father does and is forced to beg for home lessons in mathematics (p. 34). The mother character the writer invents is even more unbelievable; she is so focused on her piano that it takes 57 pages for her to realise Emmy Noether will not become a concert pianist! In the book, Ida Noether is described as good at playing, so surely one so talented would know anatomy counts for just as much as practice. Emmy is described on numerous occasions as physically awkward.

It is accepted that the fate of women in the misogynist Germany of the time was dismal to say the least [1] but what one does in public and private are two separate things. It would have made more sense and better reading if the piano lessons, cooking, sewing and languages (none of which Emmy excelled at) were simply for social expectations and she received encouragement and help from her family in the earliest years. Evenings of classical piano do not in themselves constitute a warm, intellectual environment, and since mathematics is not a 9-to-5 subject, it doesn't seem plausible to expect Emmy growing up in a mathematical vacuum. Living in an academic house promotes an atmosphere conducive to higher thinking: a fact noted by Lucy Hawkings [2] but evidently denied by Margaret Tent. Even though we cannot prove the true source of Emmy Noether's creativity, framing her in a 'woman's world' until she matures doesn't explain her brilliant mind, her sudden choice to study mathematics (instead of teaching language), the natural ease she displayed or even her drive to explore the subject and create her own algebra. If her brilliance were purely attributed to her natural propensity then we would expect her progress to be directed not comprised of a series of tentative steps and occasional retrogressions (p. 108).

As the story unfolds we are treated to instances of Noether finding her mathematical calling, where she develops her thinking away from the tedious algebra of Gordon (p. 61) to the axiomatic vision of Hilbert and that this process is ongoing such that she developed her mathematics as she taught (p. 106). Noether reached her peak of intellectual prowess through a long process with the aid of mentors, peers, and personal dedication to her craft; it is simply not an accident Noether's greatest achievements start once she's 30 years old when, after completing a PhD at 25, she discovers Dedekind through the help of her new mentor Ernest Fischer (she is 29). Margaret Tent uses this delayed emergence as proof that Emmy Noether was not a child prodigy (p. 159) yet she acknowledges the earliest fact of Emmy as a child that she solved puzzles and logic games with ease (p. x).

The purpose of the book is to detail the emergence of modern algebra. Mathematically, however, the book is dotted (sparingly) with brief and inadequate descriptions of Noether's research into rings (starting p. 61). Simple mathematical concepts (e.g. negative numbers) are painstakingly worked out with examples, but for the complicated subjects, the short paragraphs and conversations are not given the same attention to detail. The glossary is mathematically incomplete: differential, group theory, hyperbolic geometry, hypercomplex numbers, linear algebra, number theory, projective geometry and topological spaces are omitted from the list and only appear in the text as words without description. More shocking is the notable absence from the glossary of abstract algebra (the whole point of the book), commutative rings and Noetherian rings, although they are mentioned in the text. Apart from not providing a complete, useful glossary, the impact of Noether's enormous contribution is not given enough description. With the excuse of a young audience, the mathematics is avoided but clearly portraying a mathematician's life requires a more thoughtful approach to the value of their work and the consequences of their achievements.

Noether's development of modern algebra was an intellectual challenge over a period of at least 20 years made all the more difficult because of her gender. The recurring theme of the book is her disadvantage being female and a Jewess although she managed to defy the establishments by succeeding in gaining her doctorate and publishing. Although surrounded by men, she managed to surpass any barriers associated with sex. It is a shame that this awe she inspired in her male colleagues was not discussed more in the book because this is surely one of the most inspiring characteristics of Emmy Noether. Despite the prejudices, exceptions were made for those displaying promise and Emmy Noether gained support of all her professors, colleagues and students. Especially David Hilbert and Hermann Weyl, both who championed her cause of a woman's right to education, and Helmut Hasse [3] fighting to keep her in Germany in 1933. As for Hasse, I have doubts to the research conducted by Tent when she claims Hasse tried to join the Nazi party (p. 144) almost compromising his rapport with Noether, yet the documents do not correlate this.

One cannot deny Emmy Noether was a serious mathematician and it is clear not only from the book but other sources that Noether was obsessive — every moment focused on talking, writing and thinking mathematics — yet there is an avoidance by Tent to delve into the mind of Emmy through correspondence she shared with her peers. Dava Sobel explores a very human side of Galileo through the letters of his daughter [4] and Tent had access to an equally precious resource in a collection of postcards written by Emmy Noether to her mentor Ernest Fischer over an 18-year period (p. 66) providing a window into the mind of Noether (p. 76) but for their mathematical content the letters are only mentioned.

Margaret Tent denies any romantic tendencies in her heroine by giving Noether two diametrically opposed choices to make in life: marriage or mathematics. Emmy chose the sublime. There is no other mention of her loves other than mathematics, but the fact that Fischer kept her postcards (hundreds in all) suggests a deeper relationship missed by the writer. Another source of letters, those to Helmut Hasse [3], give an intriguing glimpse not only into Noether's mind but also her heart with the tone of their exchange being more than a close corroboration of mathematical ideas.

Biographies are always difficult when much of what is already written diminishes the opportunities for an original work but it is the duty of a biographer to examine the facts known of a person's life and explore the reasons for their actions based on accounts and the records left behind; not to create the story they would like to read. This book is not a biography, being fanciful and not keeping enough to the facts, and as a creative piece it also fails because the writing is poor and disjointed. The academic achievements of Noether are inadequate and her portrait appears only in a group with her brothers. There seem to be inaccuracies and some sources are dubious so I cannot recommend this book for a first reader into the life of Emmy Noether. But for those acquainted with Noether, as told in traditional accounts, the snippets involving her students (starting p. 88) are simply delightful.

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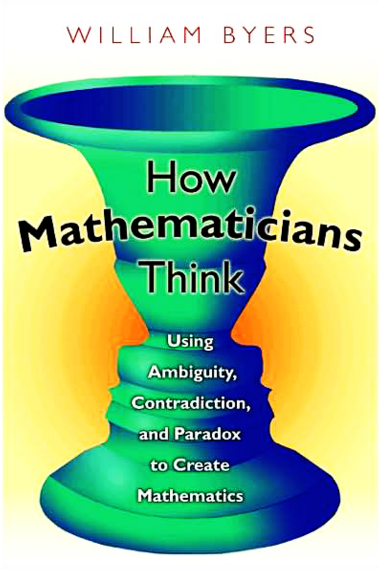
How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics

William Byers
Princeton University Press, 2007, ISBN 13: 978-0-691-12738-5

This is a rather awkward book to read. It switches, in a fairly chaotic manner, from the philosophic to the personal to the elementary to the sophisticated to the hype to the unusual to the polemic and even to the semi-religious. But to dismiss it, because of this, would be incorrect. It makes and examines, from a wide variety of perspectives, the fundamentally important point that 'mathematical creativity is a struggle with resolving ambiguity, contradiction and paradox'.

Strictly speaking, the author neither answers the question 'How mathematicians think?' nor explains the action 'How mathematicians think!'. The book is about 'how mathematicians practice and view mathematics'. What it does do is explain how mathematical creativity is initiated, stimulated and fostered by ambiguity, contradiction and paradox. As a consequence, it will stimulate awareness that the action of creating and applying mathematics, even elementary, is a dynamic activity which is different from the static nature of mathematical knowledge. For the author, known mathematical results are the small levers and pulleys which are the only tools that are available to move the current contradictory, ambiguous and paradoxical obstacles out of the way to expose new understanding and mathematical results.

What does one like to learn when one reads a book? Because the reading of a book is a union between its text and the reader's consciousness, one answer is the wedding custom of 'something old, something new, something borrowed, something blue'. All are there in this book.



The things old are reflected in the discussion of famous historical treasures of mathematics — the discovery of zero; Fermat's last theorem; Archimedes' method for estimating the value of *pi*; Zeno's paradox; the ambiguity of infinity.

The things new are the various points that the author has formulated to pursue the fundamental theme that permeates the book. For example, creativity is not based on the magical notion that it just happened, without making mistakes, but on the struggle that is undertaken in resolving ambiguity, contradiction and paradoxes. In fact, this holds, in related ways, for all forms of creativity. This point is made by the author drawing on Arthur Koesteler's definition of creativity, Zen Buddhism and music (p. 71–76).

The things borrowed are the numerous illustrations used to support the different points developed in the text — historical; popular; the views of mathematicians on a variety of issues; comments by living mathematicians (e.g. Andrew Wiles and William Thurston); perspectives from mathematical education.

The things blue are the various comments like 'This book puts forward a new vision of what mathematics is all about' (p. 12); 'Without this double or ambiguous point of view, modern mathematics would never have been invented' (p. 43).

What would one expect to learn if one decided to read a book entitled 'How Mathematicians Think'? The possibilities include:

- to gain a clear understanding of the technical details that are involved when a mathematician struggles to understand and advance mathematical knowledge;
- to be given a physiological/psychological description of the brain processes involved;
- to read a philosophic description of the subject.

In this book, it is essentially a mixture of (i) and (iii). For (ii), one must turn to Hadamard [1], which is not mentioned in this book.

Chapter 1, after an Introduction entitled 'Turning on the Light', starts with the following seminal comment of David Bohm:

> I think that people get it upside down when they say the unambiguous is the reality and the ambiguous merely uncertainty about what is really unambiguous. Let's turn it around the other way: the ambiguous is the reality and the unambiguous is merely a special case of it, where we finally manage to pin down some very special aspect.

This quotation, which crisply summarises the essential message around which the book is written and is discussed in some detail in this chapter, clearly pinpoints how mathematics is viewed by the general public as a collection of unambiguous rules compared with how it is practised and applied by mathematicians.

As the text of Chapter 1 illustrates in various ways, including a discussion of the discovery of 'zero', there is a clear lack of appreciation, in the wider community, about the intellectual struggle that has been performed historically to derive and quantify the unambiguous rules that are taught to be learnt by heart and to apply without question.

It is a useful book for the apprentice mathematician by clarifying the importance of boldness in making mistakes and declaring that one does not fully understand some technical details which at first sight appear to be more complex than they really are. Unfortunately, however, it is not a book that such mathematicians would make the effort to continue to read after working though the first few chapters. They would appreciate that good points are being made about the practice of mathematics, but would be perplexed by the way that the logic appears to jump randomly from one issue to the next.

It is unfortunate that this review is not as positive as I would have liked it to have been. There is simply no doubt about the merits of the task that the author has undertaken. Unfortunately, by trying to do too much with what is a neat, perceptive and seminal idea, the author has stolen defeat from the jaws of victory. A book, or a number of books half the size or less, would have engendered victory. This would have been of great benefit to promulgating the seminal information about the practice of mathematics to apprentice mathematicians and the wider public community.

My advice to the potential reader: pick through the contents like a chicken looks for the titbits that suit its needs. From this point of view, the book has a reasonable index, which includes the names of people specifically mentioned or quoted.

As I conclude this review, I am reminded of Hans Christian's Andersen's story about the 'Emperor's New Clothes'; the author has dressed his themes in the wrong clothes, making me feel like the child in that story.

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