Differentiability of a pathological function, diophantine approximation, and a reformulation of the Thue–Siegel–Roth theorem

Juan Luis Varona*

Abstract

We study the differentiability of the real function

$$f_\nu(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/q^\nu, & \text{if } x = p/q \in \mathbb{Q}, \text{ an irreducible fraction,} \end{cases}$$

for different values of ν. For every $\nu > 0$, the function f_ν is continuous at the irrationals and discontinuous at the rationals. But perhaps the most interesting case is what happens for $\nu > 2$. In this case, it is shown that f_ν is differentiable in a set D_ν such that both D_ν and $\mathbb{R} \setminus D_\nu$ are dense in $\mathbb{R}$. Moreover, the Lebesgue measure of the set $\mathbb{R} \setminus D_\nu$ is 0. In the proofs, the diophantine approximation by means of continued fractions is used. Finally, we show a nice reformulation of the Thue–Siegel–Roth theorem in terms of the differentiability of f_ν for $\nu > 2$.

A well-known pathological real function is

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/q, & \text{if } x = p/q \in \mathbb{Q}, \text{ an irreducible fraction,} \end{cases}$$

where, here and in the rest of the paper, we assume that, when we write a rational number p/q, we have $p, q \in \mathbb{Z}$ and $q > 0$ (in particular, $f(k) = f(k/1) = 1$ for every $k \in \mathbb{Z}$, including $k = 0$). This function is of interest because it is discontinuous at the rationals and continuous at the irrationals. For completeness, let us prove it.

If $x = p/q \in \mathbb{Q}$, let us take a sequence $\{x_n\}$ of irrational numbers such that $x_n \to x$ when $n \to \infty$; then $f(x_n) = 0$ for every n and the sequence $\{f(x_n)\}$ does not converge to $f(x) = 1/q$, so f is not continuous at x. On the other hand, for $x \in \mathbb{R} \setminus \mathbb{Q}$, let us see that f is continuous at x by checking that $f(x_n) \to f(x) = 0$ for every sequence $\{x_n\}$ that tends to x. As $f(y) = f(x)$ for every irrational number y, we can consider, without loss of generality, that $x_n = p_n/q_n \in \mathbb{Q}$ for every n. Now, from $p_n/q_n \to x$, an irrational number, it follows that $q_n \to \infty$. Then, $f(x_n) = 1/q_n \to 0 = f(x)$ and so f is continuous at x.

But, what about the differentiability of f?

Received 29 February 2008; accepted for publication 6 October 2009

*Departamento de Matemáticas y Computación, Universidad de La Rioja, Calle Luis de Ulloa s/n, 26004 Logroño, Spain. E-mail: jvarona@unirioja.es

The research of the author is supported by grant MTM2009-12740-C03-03 of the DGI

It is clear that f is not differentiable at rational numbers (it is not continuous); moreover, if the derivative exists for an irrational number, it must be zero. Thus, given x an irrational number, we only need to check whether, for arbitrary sequences $\{x_n\}$ that tend to x (where we can again assume rationals $x_n = p_n/q_n$), we always have

$$\lim_n \frac{f(x_n) - f(x)}{x_n - x} = 0 \tag{1}$$

or not.

This can be analysed in terms of the approximation of real numbers by rationals. Let us remember that, for any irrational x, there exists a positive constant C such that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{C}{q^2} \tag{2}$$

has infinitely many rational solutions p/q; this is Dirichlet's theorem, that is an easy consequence of the pigeonhole principle (moreover, Hurwitz's theorem ensures that the smallest constant for which this property is true for every irrational x is $C = 1/\sqrt{5}$). Thus, we can build a sequence of different rational numbers $\{p_n/q_n\}$ (where $q_n \to \infty$) such that $|x - p_n/q_n| < C/q_n^2$. Then,

$$\left| \frac{f(x_n) - f(x)}{x_n - x} \right| = \left| \frac{1/q_n - 0}{p_n/q_n - x} \right| > q_n/C,$$

that tends to infinity, so (1) is not satisfied, and f is nowhere differentiable.

Is it possible to build examples similar to f but in such a way that the function is differentiable in some set? Perhaps the differentiability will increase by defining $f(p/q) = 1/q^\nu$ for big values of ν? So, in this paper we are going to analyse the differentiability of the real function

$$f_\nu(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/q^\nu, & \text{if } x = p/q \in \mathbb{Q}, \text{ irreducible,} \end{cases}$$

for various values of $\nu \in \mathbb{R}$. Actually, a large proportion of this study has already been covered in the literature; see, for instance, [2], [3], [6], [7]. Here we present some results that are already known (usually with a different proof), and some that seem to be new. In the opinion of this author, f_ν is a very interesting function, and it is worthwhile to continue analysing its behaviour.

In this way, we find examples of functions whose properties about continuity and differentiability are pathological at the same time. For every $\nu > 0$, the function f_ν is continuous at the irrationals and discontinuous at the rationals. And, when $\nu > 2$ (that is the most interesting case), we prove that f_ν is differentiable in a set D_ν such that both D_ν and $\mathbb{R} \setminus D_\nu$ are dense in $\mathbb{R}$. Moreover, the Lebesgue measure of the set $\mathbb{R} \setminus D_\nu$ is 0. It is astonishing that, differentiability being a local concept, f_ν is differentiable almost everywhere in spite of the fact that it is not continuous at any rational number.

We finish the paper by showing a reformulation of the Thue–Siegel–Roth theorem in terms of the differentiability of f_ν for $\nu > 2$ (see Theorem 3 and the final Remark). It seems surprising that a theorem about diophantine approximation is equivalent to another theorem about the differentiablity of a real function: a nice new connection between number theory and analysis! As far as I know, this characterisation of the Thue–Siegel–Roth theorem has not been previously observed.

Remark 1. The pathological behaviour of functions is a useful source of examples that help to understand the rigorous definitions of the basic concepts in mathematical analysis. In this respect, it is interesting to note that, here, we have shown a kind of pathological behaviour that is different from that of the more commonly studied: the existence of continuous nowhere differentiable real functions, whose most typical example is the Weierstrass function $\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$, for $0 < a < 1$ and $ab \geq 1$; see [1] for a recent proof, or [8], [10] for a couple of surveys on this subject.

Case $\nu \leq 0$

In this case, it is clear that f_ν is nowhere continuous, so nowhere differentiable.

Case $0 < \nu \leq 2$

Now, the same proof of the case $\nu = 1$ serves to show that, when $0 < \nu \leq 2$, the function f_ν is discontinuous at the rationals, continuous at the irrationals, and nowhere differentiable. Actually, this can be found in [3] (where it is also observed that f_ν is nowhere Lipschitzian when $0 < \nu < 2$).

Case $\nu > 2$

The key to study the differentiability of f_ν at an irrational number is to analyse the diophantine approximation of such number. To obtain good rational approximations of an irrational number, one of the most used methods is to employ continued fractions, so let us briefly introduce it. See [4] or [5, Chapter 7] for details. (Although in a different way, continued fractions are also being used to study the differentiability in [6].)

For an irrational number x, let be

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{3}$$

its expansion as an (infinite) continued fraction; here $a_k \in \mathbb{Z}$ and $a_k > 0$ for $k > 0$, and they are called the elements of the continued fraction. Thus, we say that x has bounded elements if there exists a constant M such that $a_k < M$ for every k.

By truncating (3) up to a_k, we get a fraction p_k/q_k (the so-called *convergent* or *approximant* of x); these fractions can be obtained by means of the recurrence relation

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2}, \\ q_k &= a_k q_{k-1} + q_{k-2} \end{aligned}$$

starting with $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, and $q_0 = 1$. One of the basic facts of the approximation by continued fractions is that

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} \tag{4}$$

for every $k \geq 0$. From this, and taking into account that $\{q_k\}$ is always an increasing sequence, it follows that $|x - p_k/q_k| < 1/q_k^2$ for every convergent p_k/q_k. In particular, this proves that (2) with $C = 1$ has infinitely many rational solutions.

Given an irrational number x, it is not always possible to get diophantine approximation of order $\varphi(q)$ for functions $\varphi(q)$ that decrease faster than $1/q^2$. But, by constructing suitable x by means of continued fractions, this is sometimes possible. For instance, if we take $\varphi(q)$ an arbitrary positive function, let us construct a continued fraction x by choosing its elements in such a way that they will satisfy the inequalities

$$a_{k+1} > \frac{1}{q_k^2 \varphi(q_k)}, \qquad k \geq 0.$$

This, of course, can be done in an infinite number of ways; in particular, a_0 can be chosen arbitrarily. Then, from the right inequality in (4), we have

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} = \frac{1}{q_k(a_{k+1} q_k + q_{k-1})} \leq \frac{1}{a_{k+1} q_k^2} < \varphi(q_k)$$

for any $k \geq 0$. In this way, we have proved the following (note that x is irrational because the continued fraction does not terminate):

Lemma 1 ([4, Theorem 22, p. 35])**.** *For any positive function $\varphi(q)$ with natural argument q, there exist infinitely many irrational numbers x such that the inequality*

$$\left| x - \frac{p}{q} \right| < \varphi(q) \tag{5}$$

has an infinite quantity of rational solutions p/q.

Also, let us recall the following result, whose proof is also an easy consequence of (4) and other simple properties of the continued fractions:

Lemma 2 ([4, Theorem 23, p. 36])**.** *For every irrational number x with bounded elements, and for sufficient small C, the inequality*

$$\left| x - \frac{p}{q} \right| < \frac{C}{q^2} \tag{6}$$

has no rational solution p/q. On the other hand, for every number x with an unbounded sequence of elements and arbitrary $C > 0$, the inequality (6) has an infinite set of such solutions.

Then, we have the following theorem:

Theorem 1. *For $\nu > 2$, the function f_ν is discontinuous (and consequently not differentiable) at the rationals, and continuous at the irrationals. With respect to differentiability, we have:*

- (a) *For every irrational number x with bounded elements in its continued fraction expansion, f_ν is differentiable at x.*
- (b) *There exist infinitely many irrational numbers x such that f_ν is not differentiable at x.*

Moreover, the sets of numbers that fulfill (a) and (b) are both uncountable.

Proof. The continuity is treated as in the case $\nu = 1$.

(a) For irrational numbers it is clear that, if f_ν is differentiable at x, it must be $f'_\nu(x) = 0$. Let us see that this occurs for irrational numbers x with bounded elements. For that, we only need to check that, for every sequence $\{x_n\}$ that tends to x (and with $x_n \neq x$ for all n), we have

$$\lim_n \frac{f_\nu(x_n) - f_\nu(x)}{x_n - x} = 0.$$

Without loss of generality, we can assume that $\{x_n\}$ is a sequence of rationals, say $x_n = p_n/q_n$. The first part of Lemma 2 ensures that, for some value of C, we have $|x - p_n/q_n| \geq C/q_n^2$ for every n. Then,

$$\left| \frac{f_\nu(p_n/q_n) - f_\nu(x)}{p_n/q_n - x} \right| = \left| \frac{1/q_n^\nu - 0}{p_n/q_n - x} \right| \leq \frac{1/q_n^\nu}{C/q_n^2} = \frac{1}{C q_n^{\nu-2}},$$

that tends to 0 when $n \to \infty$, so f_ν is differentiable at x.

(b) Finally, let us take, in Lemma 1, $\varphi(q) = 1/q^{\nu+1}$. Then, for x such that the inequality $|x - p/q| < \varphi(q)$ has infinitely many solutions, let us take $\{p_n/q_n\}$ a sequence of rationals such that $|x - p_n/q_n| < 1/q_n^{\nu+1}$ for every n. In particular, $p_n/q_n \to x$ and verifies

$$\left| \frac{f_\nu(p_n/q_n) - f_\nu(x)}{p_n/q_n - x} \right| = \left| \frac{1/q_n^\nu - 0}{p_n/q_n - x} \right| > \frac{1/q_n^\nu}{1/q_n^{\nu+1}} = q_n,$$

that tends to ∞ when $n \to \infty$, so f_ν is not differentiable at x.

That both sets in (a) and (b) are uncountable is clear by construction of the corresponding x in Lemmas 2 and 1, respectively. (The usual diagonal argument of Cantor to show the uncountability can be used.)

In terms of the theory of measure, what is more common for the irrationals: the differentiability or the non differentiability?

For this purpose, we will use the following result (which is usually known as Khinchin's theorem). As usual, 'almost all x' means 'every x except a set of measure zero'.

Lemma 3 ([4, Theorem 32, p. 69])**.** *Suppose that $g(t)$ is a positive continuous function of a positive variable t such that $tg(t)$ is a non-increasing function. Then,* *the inequality*

$$\left| x - \frac{p}{q} \right| < \frac{g(q)}{q} \tag{7}$$

has, for almost all x, an infinite quantity of rational solutions p/q if, for some positive s, the integral

$$\int_s^\infty g(t)\,\mathrm{d}t \tag{8}$$

diverges. On the other hand, inequality (7) has, for almost all x, only a finite quantity of rational solutions p/q if the integral (8) converges.

The second part of this lemma is the main tool to prove the following theorem, that summarises the pathological behaviour of f_ν when $\nu > 2$. (Note that a different proof of the almost everywhere differentiability of f_ν, that does not use Khinchin's result, can be found in [2]. And see [3] for a proof of the density without explicitly using the measure.)

Theorem 2. *For $\nu > 2$, let us denote*

$$C_\nu = \{\, x \in \mathbb{R} \colon f_\nu \text{ is continuous at } x \},$$
$$D_\nu = \{\, x \in \mathbb{R} \colon f_\nu \text{ is differentiable at } x \}.$$

Then, the Lebesgue measure of the sets $\mathbb{R} \setminus C_\nu$ and $\mathbb{R} \setminus D_\nu$ is 0, but the four sets C_ν, $\mathbb{R} \setminus C_\nu$, D_ν, and $\mathbb{R} \setminus D_\nu$ are dense in $\mathbb{R}$.

Proof. We have $C_\nu = \mathbb{R} \setminus \mathbb{Q}$ (so $\mathbb{R} \setminus C_\nu = \mathbb{Q}$), and it is well known that $\mathbb{Q}$ has measure 0, and that both the rational and the irrational numbers are dense in the reals, i.e. $\overline{C_\nu} = \overline{\mathbb{R} \setminus C_\nu} = \mathbb{R}$. From this, and noticing that $\mathbb{R} \setminus C_\nu \subset \mathbb{R} \setminus D_\nu$, also follows $\overline{\mathbb{R} \setminus D_\nu} = \mathbb{R}$.

To compute the measure of $\mathbb{R} \setminus D_\nu$, let us take $g(t) = 1/(t \log^2(t+1))$. As $\int_1^\infty 1/(t \log^2(t+1))\,\mathrm{d}t < \infty$, the second part of Lemma 3 shows that, for almost all x, the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2 \log^2(q+1)}$$

has only a finite quantity of rational solutions. From here, it is clear that, for each x, there exists a positive constant $C(x)$ such that

$$\left| x - \frac{p}{q} \right| < \frac{C(x)}{q^2 \log^2(q+1)}$$

has no rational solution. Also, let us note that, as the rationals have measure 0, the same can be said for almost all irrational x. We claim that f_ν is differentiable at these x; consequently, the measure of $\mathbb{R} \setminus D_\nu$ is 0.

To prove the claim, let us proceed as in (a) of Theorem 1. For every sequence of rationals p_n/q_n that tends to the irrational x, we have

$$\left| x - \frac{p_n}{q_n} \right| \geq \frac{C(x)}{q_n^2 \log^2(q_n+1)}$$

for all n. Then,

$$\left| \frac{f_\nu(p_n/q_n) - f_\nu(x)}{p_n/q_n - x} \right| \le \frac{1/q_n^\nu}{C(x)/(q_n^2 \log^2(q_n+1))} = \frac{\log^2(q_n+1)}{C(x) q_n^{\nu-2}},$$

which tends to 0 when $n \to \infty$. This proves that f_ν is differentiable at x.

Finally, the denseness of D_ν follows by using that, if a set $\mathbb{R} \setminus S$ has Lebesgue measure 0, then S is dense in $\mathbb{R}$. The proof of this fact is well known, but we reproduce it for completeness. The closure of S is a closed set; if there exists a real number x that does not belong to $\overline{S}$, there exists an open interval I around x such that $I \cap \overline{S} = \emptyset$, and consequently also $I \cap S = \emptyset$ and so $I \subset \mathbb{R} \setminus S$; but I has positive measure, which is a contradiction.

Remark 2. In terms of the variation of a function, it seems natural that f_ν to be differentiable almost everywhere when $\nu > 2$. Let us recall that, in a closed interval, a real function is differentiable almost everywhere if it is of bounded variation. In any interval $[k, k+1]$ (with $k \in \mathbb{Z}$), f_ν has are two jumps of height 1 (in the extremes), a jump of height $1/2^\nu$ (in the middle point), two jumps of height $1/3^\nu$, three jumps of height $1/4^\nu$, and so on. Then, the variation of f_ν in $[k, k+1]$ is bounded by

$$2 + \sum_{q=2}^{\infty} \frac{q-1}{q^\nu},$$

which is convergent when $\nu > 2$.

The theorem of Thue–Siegel–Roth revisited

The Thue–Siegel–Roth theorem (also known simply as Roth's theorem) is a fundamental result in the field of approximation by rationals. It was proved by Roth in 1955 (he received a Fields medal for this result), and it is the final step following previous efforts by Thue, Siegel, Gelfond and Dyson through the first part of the 20th century. The original paper from Roth is [9]; but see also [5, Chapter 6], for a detailed proof.

This theorem asserts that, if x is an algebraic number, and we take an arbitrary $\alpha > 0$, the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\alpha}} \tag{9}$$

only has finitely many rational solutions p/q. Or, equivalently, if x is an irrational algebraic number, there exists a positive constant $C(x, \alpha)$ such that

$$\left| x - \frac{p}{q} \right| < \frac{C(x,\alpha)}{q^{2+\alpha}} \tag{10}$$

has no rational solution.

In the practice, this theorem is frequently used as a criterion for transcendence: if, for some $\alpha > 0$, the inequality (9) has infinitely many rational solutions, x must be a transcendental number. This criterion is much more powerful than the Liouville criterion, that was used by Liouville in 1844 to prove that $\sum_{k=1}^{\infty} 10^{-k!}$ is a transcendental number, the first number to be proven transcendental.

Now, we will see that the Thue–Siegel–Roth theorem can be reformulated in terms of the differentiability of f_ν.

Firstly, let us use it to prove the following theorem regarding the differentiability of f_ν. Actually, this part has been already done in [3] and [7].

Theorem 3. *Let $\nu > 2$. If x is an algebraic irrational number, then f_ν is differentiable at x.*

Proof. Let x be an algebraic irrational number and take $\alpha = (\nu-2)/2 > 0$. To prove the differentiability of f_ν at x it is enough to see that, for any sequence of rationals $\{p_n/q_n\}$ that tends to x, we have

$$\lim_n \frac{f_\nu(p_n/q_n) - f_\nu(x)}{p_n/q_n - x} = 0.$$

Because (10) does not have rational solutions, we have

$$\left| x - \frac{p_n}{q_n} \right| \geq \frac{C(x,\alpha)}{q_n^{2+\alpha}}$$

for every n, and consequently

$$\left| \frac{f_\nu(p_n/q_n) - f_\nu(x)}{p_n/q_n - x} \right| = \left| \frac{1/q_n^\nu - 0}{p_n/q_n - x} \right| \leq \frac{1/q_n^\nu}{C(x,\alpha)/q_n^{2+\alpha}} = \frac{1}{C(x,\alpha) q_n^{(\nu-2)/2}},$$

that tends to 0 when $n \to \infty$.

Read in a different way, this theorem says that 'if f_ν is not differentiable at x, then x is either a rational number or a transcendental number'. But, as happens with the Thue–Siegel–Roth criterion, the nondifferentiability at x only serves to detect a small proportion of the transcendental numbers. There are many transcendental numbers x for which f_ν is differentiable at x.

Remark 3. We have proved Theorem 3 by using the Thue–Siegel–Roth theorem. But we have said that it is a *reformulation*. So, let us see how to deduce the Thue–Siegel–Roth theorem from Theorem 3.

Given x algebraic and irrational, and $\nu > 2$, Theorem 3 ensures that f_ν is differentiable at x, so there exists

$$\lim_{y \to x} \frac{f_\nu(y) - f_\nu(x)}{y - x} = f'_\nu(x).$$

By approximating $y \to x$ by irrationals y, it follows that $f'_\nu(x) = 0$. Consequently, by approximating $y \to x$ by rationals, i.e. $y = p/q$, we also must have

$$\lim_{p/q \to x} \frac{f_\nu(p/q) - f_\nu(x)}{p/q - x} = \lim_{p/q \to x} \frac{1/q^\nu}{p/q - x} = 0.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{q^\nu} \leq \varepsilon \left| \frac{p}{q} - x \right|$$

when $p/q \in (x-\delta, x+\delta)$. From here, it is easy to check that the same happens for every $p/q \in \mathbb{Q}$, perhaps with a greater constant ε' in the place of ε. Thus, (10) with $\alpha = \nu - 2$ and some positive constant $C(x, \alpha) = 1/\varepsilon'$ has no rational solution, and we have obtained the Thue–Siegel–Roth theorem.

Acknowledgements

The author thanks Emilio Fernández, Mario Pérez, Francisco J. Ruiz and the referee for their valuable comments, which helped to improve this paper.

References

[1] Baouche, A. and Dubuc, S. (1992). La non-dérivabilité de la fonction de Weierstrass. *L'Enseignement Mathématique* **38,** 89–94.

[2] Darst, R.B. and Taylor, G.D. (1996). Differentiating powers of an old friend. *Amer. Math. Monthly* **103,** 415–416.

[3] Heuer, G.A. and Undergraduate Research Participation Group (1965). Funtions continuous at the irrationals and discontinuous at the irrationals. *Amer. Math. Monthly* **72,** 370–373.

[4] Khinchin, A.Ya. (1997). *Continued Fractions.* Dover Publications.

[5] Miller, S.J. and Takloo-Bighash, R. (2006). *An invitation to modern number theory.* Princeton University Press.

[6] Norton, A. (1988). Continued fractions and differentiability of functions. *Amer. Math. Monthly* **95,** 639–643.

[7] Nymann, J.E. (1969). An application of diophantine approximation, *Amer. Math. Monthly* **76,** 668–671.

[8] Pinkus, A. (2000). Weierstrass and approximation theory. *J. Approx. Theory* **107,** 1–66.

[9] Roth, K.F. (1955). Rational approximation of algebraic numbers. *Mathematika* **2,** 1–20; Corrigendum, *Mathematika* **2,** 168.

[10] Thim, J. (2003). *Continuous Nowhere Differentiable Functions.* Master Thesis, Luleå University of Technology (Sweden).
Available at http://epubl.ltu.se/1402-1617/2003/320/index-en.html