Square products of punctured sequences of factorials

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Abstract

The following problem is solved: for a positive integer n, for what m is $(\prod_{k=1}^{n} k!)/m!$ a perfect square ($1 \le m \le n$)?

The following appeared in Puzzle Corner 11 [7, p. 13] of a recent issue of this *Gazette*.

> **Factorial fun**
>
> The numbers $1!, 2!, 3!, \ldots, 100!$ are written on a blackboard. Is it possible to erase one of the numbers so that the product of the remaining 99 numbers is a perfect square?

The solution is given in [8, p. 178]. The problem has appeared elsewhere, for instance (as Norman Do, editor of The Puzzle Corner, informed us) as a contest problem. We found it in the 17th 'Tournament of Towns' (Spring 1996, Problem #3, O-Level for Seniors, contributed by S. Tokarev) [9].

In such a contest, using a computer to tackle the problem is likely to be grounds for disqualification. But if one is not bound by such rules or sensibilities and is unable to resist writing a quick line of code, the puzzle's answer is available quickly: 'Yes, erase the number 50! to get a perfect square'. However, having cheated in this way, one is then obliged to give a proper solution, and to generalise or otherwise make amends. We submit this note as such a gesture.

Upon seeing the answer above, one is immediately tempted to make this conjecture: with arbitrary even n instead of 100, the answer will be 'Yes, erase $(n/2)!$'. But this is false, since for $n = 98$, we have to 'erase' the number 50!, not 49!, while for $n = 102$, no such erasable number exists! Something funny is going on here. This much is true: If $4 \mid n$, then the $n/2$-conjecture holds, as we will prove below. But it is still more interesting than that, since for $n = 96$ one can erase either 48! or 49! to obtain a perfect square. Furthermore, it appears that there is no solution for odd n.

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Shortly, we will prove a few Fun Factorial Facts (FFF's) that allow us to completely solve the more general

> **Factorial Fun Problem**
> Let n be a positive integer. For which m is $\frac{1}{m!}\prod_{k=1}^{n} k!$ a perfect square $(1 \le m \le n)$?

We will let FFP(n) refer to the problem above for specific values of n and we will write $m \in \mathcal{F}(n)$ if m is one of the solutions to FFP(n). The answer to the originally stated FFP(100) is that $\mathcal{F}(100) \neq \emptyset$, since $50 \in \mathcal{F}(100)$. This is proved in Fun Factorial Fact 1 below.

Letting $n\$:= \prod_{k=1}^{n} k!$ (a quantity occasionally called the *superfactorial* of n, though this term sometimes has other meanings) and letting $\square$ denote the set of positive square integers, FFP(n) asks whether $n\$/m! \in \square$ for some $1 \le m \le n$. However, if y is a factor of x, then $x/y \in \square \iff xy \in \square$, so $m \in \mathcal{F}(n)$ if and only if $n\$m! \in \square$. Therefore, we will use this formulation of the problem from now on, since multiplying is more convenient than dividing:

FFP(n). For which m $(1 \le m \le n)$ is $n\$\,m! \in \square$?

We will answer the original problem in the affirmative by proving that if k is a positive integer, then $(4k)\$(2k)! \in \square$, or equivalently, $2k \in \mathcal{F}(4k)$.

All of our variables represent positive integers unless otherwise stated.

FFF 1. $2k \in \mathcal{F}(4k)$ for each k.

The proof of FFF 1 will be given momentarily. Now $x \in \square$ if and only if all of the prime factors of x occur an even number of times. That is, if we write the prime factorisation of x as

$$x = \prod_{p \in \mathbb{P}} p^{e_p},$$

where $\mathbb{P}$ denotes the set of prime numbers, then each power e_p is even when $x \in \square$. It will be handy to let $m \bmod \square$ denote the *square-free part* of x, by which we mean x divided by the largest square divisor of x, or equivalently,

$$x \bmod \square = \prod_{p \in \mathbb{P}} p^{e_p \bmod 2}.$$

For example, $360 = (2^3)(3^2)(5^1)$, so $360 \bmod \square = (2^1)(3^0)(5^1) = 10$. (The use of the term is not uniform; others define the square-free part of x to be $\prod\{p\colon p \in \mathbb{P}, p \mid x\}$.) Let us write $x \overset{\square}{\equiv} y$ whenever $x \bmod \square = y \bmod \square$, that is, whenever x and y have the same square-free parts. For example, $24 \overset{\square}{\equiv} 150$, since $24 \bmod \square = 150 \bmod \square = 6$. It is easy to see that $\overset{\square}{\equiv}$ is an equivalence relation. Also clear are the following two properties, of which we will make frequent use: (i) $x \in \square$ if and only if $x \overset{\square}{\equiv} 1$; (ii) if $x \in \square$, then $y \in \square$ if and only if $y \overset{\square}{\equiv} x$.

Lemma 1. If n is even, then $n\$ \overset{\square}{\equiv} 2 \cdot 4 \cdot 6 \cdots n$. If n is odd, $n\$ \overset{\square}{\equiv} 1 \cdot 3 \cdot 5 \cdots n$.

Proof.

$$\begin{aligned} n\$ &= \prod_{k=1}^{n} k! \\ &= n(n-1)^2(n-2)^3 \cdots (3)^{n-2} 2^{n-1} 1^n \\ &\overset{\square}{\equiv} n(n-2)(n-4) \cdots , \end{aligned}$$

with the last sequence of factors terminating at 1 or 2. ■

We note that for n even, we have $n\$ \overset{\square}{\equiv} 2^{n/2}(\frac{n}{2})!$. This observation facilitates the

Proof of FFF1. $(2k)!\,(4k)\$ \overset{\square}{\equiv} (2k)!\,(2k)!\,2^{2k} \in \square$. ■

There are still more solutions to FFP(n) when $4 \mid n$, and these are given by (1) in FFF 2, which follows. All the remaining solutions (for even n but $4 \nmid n$) are given in FFF 3 and (2) and (3) of FFF 2.

FFF 2.

$$k^2 - 1 \in \mathcal{F}(2k^2) \qquad \text{if } k \text{ is even,} \tag{1}$$

$$k^2 \in \mathcal{F}(2k^2 - 2) \qquad \text{if } k \text{ is odd,} \tag{2}$$

$$2k^2 \in \mathcal{F}(4k^2 - 2) \qquad \text{for all } k. \tag{3}$$

Proof. Three easy applications of the even part of the Lemma are all that is needed. For (1) we assume k is even to get

$$(k^2-1)!\,(2k^2)\$ \overset{\square}{\equiv} (k^2-1)!\,(k^2)!\,2^{k^2} = (k^2-1)!\,k^2(k^2-1)!\,2^{k^2} \in \square.$$

When k is odd,

$$(k^2)!\,(2k^2-2)\$ \overset{\square}{\equiv} (k^2)!\,(k^2-1)!\,2^{k^2-1} = k^2(k^2-1)!\,(k^2-1)!\,2^{k^2-1} \in \square,$$

which proves (2). Lastly, for any k,

$$(2k^2)!\,(4k^2-2)\$ \overset{\square}{\equiv} (2k^2)!\,(2k^2-1)!\,2^{2k^2-1} = k^2((2k^2-1)!)^2 2^{2k^2} \in \square$$

proving (3). ■

And there is still more factorial fun to be had, since the results of FFF 1 and FFF 2 do not exhaust the set of solutions to the FFP. The only ones that remain are described by

FFF 3. For odd k and $2(k^2-1) \in \square$,

$$k^2 - 2 \in \mathcal{F}(2k^2), \tag{4}$$

$$k^2 \in \mathcal{F}(2k^2 - 4). \tag{5}$$

We omit the proof of FFF 3, as it is similar to that of FFF 2. The (infinitely many) odd numbers k for which $2(k^2 - 1) \in \Box$ can be found by solving a Pell-type equation. The result is a sequence $(k_i)_{i=1}^{\infty}$, given by $k_1 = 3$, $k_2 = 17$ and $k_i = 6k_{i-1} - k_{i-2}$ for $i \geq 2$. We omit the proof of that, too, since we are mainly interested in the characterisation of the solutions as already given.

The real problem now is to prove that FFFs 1–3 give *all* the solutions to the FFP. One glaring sub-problem is to prove that there are no odd n for which a solution exists to the FFP, that is, that $\mathcal{F}(n) = \emptyset$ when n is odd. We will solve that in FFF 5. In FFF 6 we show that FFFs 1–3 give all of the solutions of FFP(n) for even n. First we give an easy negative result, which does not apply to the FFP except in the trivial cases where $m = 1$ or $m = n$, but nevertheless answers an obvious question. And its proof leads the way to proving FFF 5.

FFF 4. For each $n > 1$, $n\$ \notin \Box$.

We warm up the proof by recalling the well-known fact that $n! \notin \Box$ when $n > 1$, the easiest proof of which goes like this: The result is obvious when $n = 2$. By Bertrand's postulate (aka Chebyshev's theorem, a celebrated proof of which is due to a 19-year-old Erdős [3], [1]), there always exists a prime between m and $2m$ for $m > 1$. For even $n > 2$, there is then a prime between $n/2$ and n. That prime appears exactly once as a divisor of $n!$, so $n!$ cannot be a square. Likewise, if n is odd, there is a prime between $(n+1)/2$ and $n+1$, leading to the same result. (In fact, stronger results are known, such as the gem [4], the title of which tells all. See [2] for more serious factorial fun.) In view of our Lemma, the same idea can next be used to easily show that $n\$$ cannot be a square.

Proof of FFF 4. When $4 \mid n$, we have $2^{n/2} \in \Box$, so $n\$ \overset{\Box}{\equiv} (n/2)!$. But $(n/2)! \notin \Box$, since $n \geq 2$. When n is even but $4 \nmid n$, we have $n\$ \overset{\Box}{\equiv} 2(n/2)!$. But Bertrand & Co. imply that there is a prime between $n/4$ and $n/2$ for $n/4 \geq 2$. This handles the cases $n = 10, 14, 18, \ldots$, while the remaining cases $n = 2$ and 6 can be checked individually. Finally, for $n > 1$ and odd, we have $n\$ \overset{\Box}{\equiv} 1 \cdot 3 \cdot 5 \cdots (n-2)n$. Again, there is a prime in $(\frac{n+1}{2}, n+1)$, that is, in $[\frac{n+3}{2}, n]$. But since $\frac{n+3}{2} \geq 3$, that prime will be odd, and will appear in the product $1 \cdot 3 \cdot 5 \cdots (n-2)\, n$ exactly once. ∎

Bertrand's postulate made our preceding job easy, but for what follows we need an improvement — something similar yet stronger. Such a thing exists ([6]); we dub it:

Nagura's Improvement. *Let* $a_1 = 2, a_2 = 8, a_3 = 9, a_4 = 24, a_5 = 25$*, let* $j \in \{1, 2, 3, 4, 5\}$*, and let* x *be a real number with* $x \geq a_j$*. Then there exists at least one prime number* p *such that* $x < p < (j+1)x/j$*.*

Thus the case $j = 1$ gives Bertrand's postulate, while taking $j = 5$ gives the statement that there is always a prime between x and $6x/5$ when $x \geq 25$.

FFF 5. If n is odd, then $\mathcal{F}(n) = \emptyset$.

Proof. With n odd we have $n\$m! \overset{\square}{\equiv} (1 \cdot 3 \cdot 5 \cdots n)m!$. We will first assume that $n \geq 72$. (We will address smaller n later.) Using Nagura's improvement with $j = 5$, there exists a prime $p \in (\frac{5n}{6}, n)$ if $5n/6 \geq 25$, that is, $n \geq 30$. Therefore, if $m < p$, then $m \notin \mathcal{F}(n)$, for p will occur exactly once in the prime factorisation of $(1 \cdot 3 \cdot 5 \cdots n)\, m!$. Thus we may assume $m \geq p > 5n/6$. By 'pairing away' duplicate factors we can now write, for even m,

$$n\$\, m! \overset{\square}{\equiv} 2^{m/2} \left[1 \cdot 2 \cdot 3 \cdots \frac{m}{2}\right] \cdot [(m+1)(m+3) \cdots (n-2)n], \tag{6}$$

and if m is odd we use

$$n\$\, m! \overset{\square}{\equiv} 2^{(m-1)/2} \left[1 \cdot 2 \cdot 3 \cdots \frac{m-1}{2}\right] \cdot [(m+2)(m+4) \cdots (n-2)n].$$

We note the gap in each sequence of factors. We now show the details for the case of even m, the odd case being similar.

Again using Nagura's improvement, there exists a prime $q \in (\frac{2m}{5}, \frac{m}{2})$ whenever $2m/5 \geq 24$, or $m \geq 60$. This condition is met, since $m > 5n/6$ and $n \geq 72$. We claim that q occurs exactly once in the prime factorisation of the right-hand side of (6). Clearly $q \in [1, m/2]$, so q occurs among the leftmost bracketed group of explicit factors in (6). Therefore, if there is another factor of q in (6), there must be a multiple of q hiding in one of those bracketed groups. (Since $q > 2$, we need not worry about the factor $2^{m/2}$.) We first note that $2q > m/2$, so $2q$ is not in the sequence within the leftmost brackets, and since there are no even numbers greater than $m/2$ in (6), $2q$ does not occur in the rightmost brackets. Finally, any higher multiple of q is $\geq 3q > 6m/5 \geq 6p/5 > n$, so cannot be any of those explicit factors in (6). Therefore there is exactly one factor of the prime q in (6), proving $n\$\, m! \notin \square$, at least for $n \geq 72$. The cases $n < 72$ can be checked by hand (or by computer; we have a *Mathematica* program that checked all cases of $n \leq 17\,000$).

Not much changes if m is odd, and we leave those details to the diligent reader. ■

For the rest of the solutions when n is even, the situation is a bit easier but there are more cases to check.

FFF 6. If n is even, then the solutions to FFP(n) are precisely those given in FFFs 1–3.

Proof. Case 1: $4 \mid n$. Using the Lemma and then pairing away common terms in the factorial sequences, we have

$$\begin{aligned} n\$\, m! &\overset{\square}{\equiv} 2^{n/2} \left(\frac{n}{2}\right)!\, m! \\ &\overset{\square}{\equiv} \begin{cases} (m+1)(m+2) \cdots (n/2) & \text{if } m < n/2, \\ 1 & \text{if } m = n/2, \\ (n/2+1) \cdots (m-1)m & \text{if } m > n/2. \end{cases} \end{aligned}$$

The case $m = n/2$ gives precisely the solutions stated in FFF 1. No product of any sequence of length > 1 can be a square (see [4]), so the only way the two remaining cases give solutions is when there is really only one term in the 'sequence' and it is square, i.e. when $m+1=n/2 \in \square$ or $n/2+1=m \in \square$. Denote the square by k^2. In the event that $m+1=n/2=k^2$, we have $n=2k^2$ and $m=k^2-1$, which is the case of (1) in FFF 2. If $n/2+1=m=k^2$, we have $n=2k^2-2$ and $m=k^2$, which is the situation in (2).

Case 2: n is even but $4 \nmid n$, that is, $n \equiv 2 \pmod 4$. This time we have an odd power of 2, which gives

$$n\$\, m! \overset{\square}{\equiv} 2^{n/2}\left(\frac{n}{2}\right)! m!$$

$$\overset{\square}{\equiv} \begin{cases} 2(m+1)(m+2)\cdots(n/2) & \text{if } m < n/2, \\ 2 & \text{if } m = n/2, \\ 2(n/2+1)\cdots(m-1)m & \text{if } m > n/2. \end{cases}$$

It is therefore clear that $n\$\, m! \notin \square$ if $m = n/2$. For the other two cases, we need a somewhat stronger result, and we find just the ticket in [5], namely:

$$j > 2 \text{ and } b > 0 \;\Rightarrow\; 2\prod_{i=1}^{j}(b+i) \notin \square.$$

That is, twice a product of three or more consecutive positive integers is never a square. This implies that the sequences of consecutive integers indicated in case 2 must have length one (call this subcase 2a) or two (subcase 2b).

Subcase 2a: $m = n/2-1$ or $m = n/2+1$. This gives us, respectively, $n\$\, m! \in \square$ if and only if either $2(m+1) = 2(n/2) = r^2$ for some r, or $2(n/2+1) = 2m = r^2$ for some r. In the first of these subsubcases, we have $n = r^2$, where $n = 4k-2$ for some k (since $n \equiv 2 \pmod 4$), and this is impossible, since squares are always $\equiv 0$ or $1 \pmod 4$. In the second, we get $n = r^2-2$ and $m = r^2/2$. Letting $r = 2k$, this amounts to $n = 4k^2-2$ and $m = 2k^2$, which is encoded in (3) of FFF 2.

Subcase 2b: $m = n/2-2$ or $m = n/2+2$. Let's call these subsubcases (i) and (ii), respectively. Note that m must be odd here, since $n \equiv 2 \pmod 4$.

First consider 2b(i), where we have $n\$\, m! \in \square$ if and only if $2(m+1)(m+2) = 2(n/2-1)(n/2) = r^2$ for some r. Examining the parities involved, it must be that $2(m+1)$ is an even square and $m+2$ an odd square. (If q is an odd prime dividing $2(m+1)(m+2)$, then q cannot simultaneously divide $m+1$ and $m+2$.) Let $m+2 = k^2$, where k is odd. This implies that $r^2 = 2(m+1)(m+2) = 2(k^2-1)(k^2)$, hence $2(k^2-1) \in \square$ with k odd. (As we mentioned earlier, this has infinitely many solutions in k.) The condition $m = n/2-2$ then implies that $m = k^2-2$ and $n = 2k^2$. These give the precise description of (4) in FFF 3.

Finally, consider subsubcase 2b(ii). Here we have $n\$\, m! \in \square$ if and only if $2m(m-1) = 2(n/2+1)(n/2+2) = r^2$ for some r. Now we must have $2(m-1)$ an even square with m an odd square. Writing $r^2 = 2k^2(k^2-1)$ with k odd, we get $2(k^2-1) \in \square$ with $n = 2k^2-4$ and $m = k^2$, which is precisely the case specified in (5) of FFF 3. ■

With that, our FFP is solved.

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