A generalised converse mean value theorem

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Abstract

We present a converse of the mean value theorem for functions defined on arbitrary normed spaces.

Introduction

Given a differentiable function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ and a point $c \in I$, are there real numbers $a, b \in I$ such that $c \in]a, b[$ and $f(b) - f(a) = f'(c)(b - a)$? A simple example shows that the converse of the mean value theorem may fail. For the function $f(x) = x^3$, $x \in [-1, 1]$ and $c = 0$, we have $f'(0) = 0$ yet f is one-to-one. Sufficient conditions for the above converse to hold are established in [1].

Theorem 1 ([1]). *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function and differentiable in $]a, b[$. Given $c \in\,]a, b[$, let $k_0 > 0$ be such that $]c - k_0, c + k_0[\; \subseteq\;]a, b[$. If, for all $k \in\,]0, k_0[$,*

1. *$f'(c - k) < f'(c) < f'(c + k)$ then there exist $a_1, b_1 \in\,]a, b[$ with $c \in\,]a_1, b_1[$ and $f(b_1) - f(a_1) = f'(c)(b_1 - a_1)$.*
2. *$f'(c - k) \leq f'(c) \leq f'(c + k)$ then there exist $a_1, b_1 \in\,]a, b[$ with $c \in [a_1, b_1]$ and $f(b_1) - f(a_1) = f'(c)(b_1 - a_1)$.*

A method to compute a_1 and b_1 is given in [2]. Both papers concern real valued functions of a real variable.

Below we shall generalise Theorem 1 for real functions defined on open sets of normed spaces.

A generalisation of the converse of the mean value theorem

Let E be a normed space and for each $x, y \in E$, as usual, $[x, y]$ denotes the closed line segment with endpoints x and y.

Let U be an open set of E, $x \in U$ and F a normed space. Given a function $f\colon U \to F$, we say that f is differentiable at x if there exists a continuous linear

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map $Df_x\colon E \to F$ such that

$$f(x+h) - f(x) = Df_x(h) + |h|\phi(h),$$

where ϕ is such that

$$\lim_{h\to 0} \phi(h) = 0.$$

Theorem 2. *Let U be an open subset of E, $f\colon U \subseteq E \to \mathbb{R}$ be a C^1 function and $c \in U$ and $v \in E$ be such that*

$$[c-v, c+v] \subseteq U \tag{1}$$

$$\text{for all } t \in \,]0,1] \qquad Df_{c-tv}(v) \le Df_c(v) \le Df_{c+tv}(v). \tag{2}$$

Then there exist $a, b \in U$ satisfying

$$f(b) - f(a) = Df_c(b-a) \quad \text{and} \quad c \in [a,b].$$

Proof. Let us fix $\epsilon, k \in \mathbb{R}$ such that $0 < \epsilon < k < 1$. Define

$$L := \frac{f(c+(\epsilon-k)v) - f(c+kv)}{\epsilon - 2k} \in \mathbb{R}.$$

First case: Suppose

$$L \ge Df_c(v).$$

Define a continuous function $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) := \frac{f(c+(\epsilon-k)v) - f(c+tkv)}{\epsilon-(1+t)k}.$$

Note that

$$g(1) = L \ge Df_c(v).$$

By the mean value theorem, for each $t \in [0,1]$, there exists $d(t) \in [c+(\epsilon-k)v, c+tkv]$ such that

$$f(c+(\epsilon-k)v) - f(c+tkv) = Df_{d(t)}((\epsilon-(1+t)k)v).$$

Thus, for some $d(0) \in [c+(\epsilon-k)v, c]$,

$$g(0) = Df_{d(0)}(v).$$

If $d(0) = c$, then

$$f(c+(\epsilon-k)v) - f(c) = Df_c((\epsilon-k)v)$$

and the theorem is proved for $a = c$ and $b = c+(\epsilon-k)v$. If that is not the case, that is $d(0) \in [c+(\epsilon-k)v, c[$, by the hypothesis of the theorem, we have $g(0) \le Df_c(v)$. Then the existence of $t \in [0,1]$ satisfying the condition $g(t) = Df_c(v)$ follows from the intermediate value theorem, and

$$f(c+(\epsilon-k)v) - f(c+tkv) = Df_c((\epsilon-(1+t)k)v).$$

In this case we take $a = c+tkv$ and $b = c+(\epsilon-k)v$.

Second case: $L < Df_c(v)$.

Now define the continuous function $h : [\epsilon/k, 1] \to \mathbb{R}$ by

$$h(t) := \frac{f(c+(\epsilon-tk)v) - f(c+kv)}{\epsilon-(1+t)k}.$$

For each $t \in [\epsilon/k, 1]$, there exists $d(t) \in [c + (\epsilon - tk)v, c + kv]$ satisfying

$$f(c + (\epsilon - tk)v) - f(c + kv) = Df_{d(t)}((\epsilon - (1 + t)k)v).$$

Then $h(1) < Df_c(v)$ and $h(\epsilon/k) = Df_{d(\epsilon/k)}(v)$, for some $d(\epsilon/k) \in [c, c + kv]$. If $d(\epsilon/k) = c$, then

$$f(c + kv) - f(c) = Df_c(kv)$$

and we choose $a = c$ and $b = c + kv$. If $d(\epsilon/k) \in]c, c + kv]$ then $h(\epsilon/k) \geq Df_c(v)$. Again, by the intermediate value theorem, there exists some $t \in [\epsilon/k, 1[$ with $h(t) = Df_c(v)$, that is

$$f(c + (\epsilon - tk)v) - f(c + kv) = Df_c((\epsilon - (1 + t)k)v)$$

and in this case we take $a = c + kv$ and $b = c + (\epsilon - tk)v$.

It is obvious that the last theorem holds if we replace condition (2) by

$$\text{for all } t \in]0, 1] \qquad Df_{c+tv}(v) \leq Df_c(v) \leq Df_{c-tv}(v).$$

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