A geometric approach to saddle points of surfaces

Sudhir R. Ghorpade* ** and Balmohan V. Limaye* ***

Abstract

We outline an alternative approach to the geometric notion of a saddle point for real-valued functions of two real variables. It is argued that our treatment is more natural than the usual treatment of this topic in standard texts on calculus.

1. Introduction

What is a saddle point of a surface in 3-space? A reasonable answer is: a saddle point is like the centre point of a horse saddle or the low point of a ridge joining two peaks. In other words, a saddle point is that peculiar point on the surface which is at once a peak along a path on the surface and a dip along another path on the surface. Another answer that is mundane but more likely to fetch points in a Calculus test is as follows. A *saddle point* of a real-valued function of two real variables is a critical point (that is, a point where the gradient vanishes) which is not a local extremum. The first answer gives an intuitive description of a saddle point, while the second is the mathematical definition commonly given in most texts on Calculus. (See, e.g. [1, §9.9] or [6, §3.3].) A typical example is the hyperbolic paraboloid given by $z = xy$ or by (the graph of) the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := xy$. Here the origin is a saddle point. Indeed if we look at the paths along the diagonal lines $y = -x$ and $y = x$ in the plane, then we readily notice that the origin is at once a peak and a dip. Also, the origin is the only critical point of f and clearly f does not have a local extremum at the origin.

The aim of this paper is first, to point out that there is a significant disparity between the two answers, and second, to suggest an alternative approach to saddle points which may take care of this. The first point is easy to illustrate. There are surfaces or rather, functions of two variables where the conditions in the second answer are met but the geometric picture is nowhere close to the description in the first answer. For example, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) := x^3$ or by $f(x, y) := x^2 + y^3$, then the origin is a saddle point according to the usual mathematical definition, but the corresponding surface (Figure 1) hardly looks like a saddle that you might want to put on a horse for any rider! Another unsatisfactory aspect is

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*Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India.
**E-mail: srg@math.iitb.ac.in
***E-mail: bvl@math.iitb.ac.in

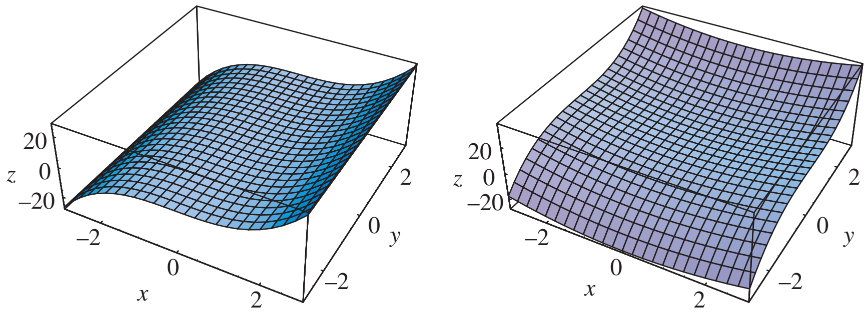

Figure 1. Graphs of $z = x^3$ and $z = x^2 + y^3$ near the origin.

the *a priori* assumption that the saddle point is a critical point, that is, a point at which the gradient exists and is zero. This is quite unlike the usual definitions of analogous concepts in one variable calculus, such as local extrema or points of inflection, where one makes a clear distinction between a geometric concept and its analytic characterisation (See, for instance, [3] and its review [8].) The definition we propose here seems to fare better on these counts in the case of functions of two variables. The basic idea is quite simple and, we expect, scarcely novel. However, we have not seen in the literature an exposition along the lines given here. For this reason, and with the hope that the treatment suggested here could become standard, we provide a fairly detailed discussion of the definition, basic results and a number of examples in the next three sections. Alternative approaches and extensions are briefly indicated in a remark at the end of the paper and we thank the referee for some of the suggestions therein.

2. Definition of a saddle point

Let D be a subset of $\mathbb{R}^2$. A *path* in D is a continuous map from $[a, b]$ to D. Here, and hereafter, while writing open or closed intervals such as (a, b) or $[a, b]$, it is tacitly assumed that $a, b \in \mathbb{R}$ with $a < b$. Given any $\boldsymbol{p} \in D$, a path $\gamma\colon [a, b] \to D$ is said to *pass through* $\boldsymbol{p}$ if $\gamma(t_0) = \boldsymbol{p}$ for some $t_0 \in (a, b)$. A path $\gamma\colon [a, b] \to D$ is said to be *regular* if γ is differentiable on (a, b) and $\gamma'(t) \neq \mathbf{0}$ for all $t \in (a, b)$. Two regular paths $\gamma_1\colon [a_1, b_1] \to D$ and $\gamma_2\colon [a_2, b_2] \to D$ are said to *intersect transversally* at some $\boldsymbol{p} \in D$ if there are $t_i \in (a_i, b_i)$ such that $\gamma_i(t_i) = \boldsymbol{p}$ for $i = 1, 2$ and moreover, $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ are not multiples of each other. In other words, the two paths pass through $\boldsymbol{p}$ and their tangent vectors at $\boldsymbol{p}$ are not parallel.

Example 1. (i) $\gamma\colon [-1, 1] \to \mathbb{R}^2$ defined by $\gamma(t) := (t, t^2)$ is a regular path, while $\widetilde{\gamma}\colon [-1, 1] \to \mathbb{R}^2$ defined by $\widetilde{\gamma}(t) := (t^2, t^3)$ is not a regular path.

(ii) If $\gamma_1, \gamma_2\colon [-1, 1] \to \mathbb{R}^2$ are defined by $\gamma_1(t) := (t, -t)$ and $\gamma_2(t) := (t, t)$, then γ_1 and γ_2 are regular paths in $\mathbb{R}^2$ which intersect transversally at the origin. Further, the path $\gamma_3\colon [-1, 1] \to \mathbb{R}^2$ defined by $\gamma_3(t) := (2t + t^2, 2t - t^2)$, is also regular and passes through the origin. The paths γ_1 and γ_3 intersect transversally at the origin, whereas the paths γ_2 and γ_3 do not.

Let $D \subseteq \mathbb{R}^2$, $\boldsymbol{p} \in D$ and $\gamma \colon [a, b] \to D$ be a regular path in D passing through $\boldsymbol{p}$ so that $\gamma(t_0) = \boldsymbol{p}$ for some $t_0 \in (a, b)$. Now, any $f \colon D \to \mathbb{R}$ can be restricted to (the image of) γ so as to obtain a real-valued function of one variable $\phi \colon [a, b] \to \mathbb{R}$ defined by $\phi(t) := f(\gamma(t))$. We shall say that f has a local maximum (or, similarly, a local minimum) at $\boldsymbol{p}$ along γ if ϕ has a local maximum (or, similarly, a local minimum) at t_0.

Definition 1. Let $D \subseteq \mathbb{R}^2$ and $\boldsymbol{p}$ be an interior point of D. A real-valued function $f \colon D \to \mathbb{R}$ has a *saddle point* at $\boldsymbol{p}$ if there are regular paths γ_1 and γ_2 in D intersecting transversally at $\boldsymbol{p}$ such that f has a local maximum at $\boldsymbol{p}$ along γ_1, while f has a local minimum at $\boldsymbol{p}$ along γ_2.

The above definition is a faithful abstraction of the idea that a saddle point is the point at which the graph of the function is at once a peak along a path and a dip along another path. The condition that the two paths intersect transversally might seem technical. But its significance will be clear from Example 2(iii) below.

It may be remarked that in our definition of a saddle point, we have permitted ourselves as much laxity as is usual while defining local extrema. To wit, if a function is locally constant at $\boldsymbol{p}$, then it has a local maximum as well as a local minimum at $\boldsymbol{p}$. In the same vein, a locally constant function at $\boldsymbol{p}$ has a saddle point at $\boldsymbol{p}$. More generally, if a function is locally constant along two regular paths intersecting transversally at $\boldsymbol{p}$, then it has a saddle point at $\boldsymbol{p}$. If we don't want to be so indulgent, then we can use the stronger notion of a *strict* saddle point. A *strict saddle point* is defined simply by replacing in Definition 1, local maximum by strict local maximum and local minimum by strict local minimum. Indeed, it is the notion of a strict saddle point that comes closest to our geometric intuition about saddle points. In almost all the examples as well as the criteria for saddle points discussed here, it is seen that the function has, in fact, a strict saddle point.

Example 2. (i) [Hyperbolic paraboloid] The function $f \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := xy$ has a saddle point at $(0, 0)$. To see this, it suffices to consider the paths γ_1 and γ_2 in Example 1(ii). Similarly, one can see that if $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$, then $f \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := (ax + by)(cx + dy)$ has a saddle point — in fact, a strict saddle point, at $(0, 0)$.

(ii) [Monkey saddle] The function $f \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x^3 - 3xy^2$ has a strict saddle point at the origin. To prove this, it helps to look at the level curves of f. We then find that it suffices to consider the parabolic paths given by $t \mapsto (-t\sqrt{3} + t^2,\ t + t^2\sqrt{3})$ and $t \mapsto (t\sqrt{3} - t^2,\ t + t^2\sqrt{3})$ for $t \in [-\sqrt{3}, \sqrt{3}]$. The surface $z = f(x, y)$ or the graph of f near the origin is shown in Figure 2 on the left. It may be interesting to try and visualise these paths on this surface.

(iii) [Fake saddle] Consider $f \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x^3$. In this case $\gamma_1, \gamma_2 \colon [-1, 1] \to \mathbb{R}$ defined by $\gamma_1(t) := (-t^2, t)$ and $\gamma_2(t) = (t^2, t)$ are regular paths passing through $\mathbf{0}$. Also, f has a strict local maximum at $\mathbf{0}$ along γ_1 and a strict local minimum at $\mathbf{0}$ along γ_2. However, γ_1 and γ_2 do not intersect transversally at $\mathbf{0}$. In fact, as the surface on the left in Figure 1 indicates, f does not have a saddle point at $\mathbf{0}$. A formal proof of this is given later in Example 4(iii).

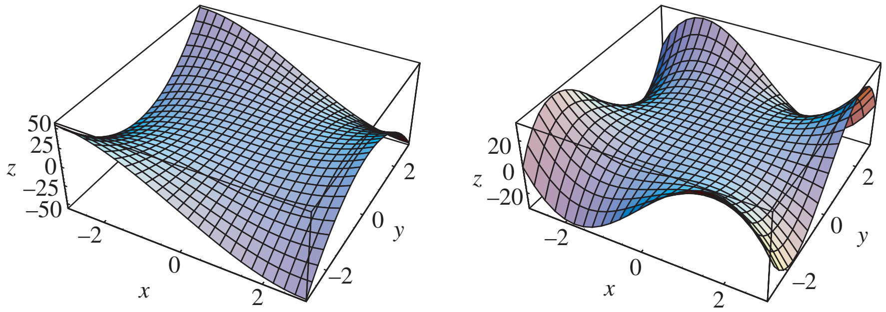

Figure 2. Monkey saddle $z = x^3 - 3xy^2$ and dog saddle $z = x^3y - xy^3$

We now show that a saddle point is necessarily a critical point. In what follows, by $\nabla f(\boldsymbol{p})$ we denote the gradient of a function f at an interior point $\boldsymbol{p}$ of its domain.

Proposition 1. *Let $D \subseteq \mathbb{R}^2$ and $\boldsymbol{p}$ be an interior point of D. If $f\colon D \to \mathbb{R}$ is differentiable at $\boldsymbol{p}$ and has a saddle point at $\boldsymbol{p}$, then $\nabla f(\boldsymbol{p}) = \mathbf{0}$.*

Proof. For $i = 1, 2$, let $\gamma_i\colon [a_i, b_i] \to D$ satisfy the conditions in Definition 1 with $n = 2$, and let us write $\boldsymbol{p} = \gamma_i(t_i)$ with $t_i \in (a_i, b_i)$ and $\phi_i := f \circ \gamma_i$. Since f is differentiable at $\boldsymbol{p}$ and γ_i is regular, by the chain rule, $\phi_i'(t_i)$ exists and equals $\nabla f(\boldsymbol{p}) \cdot \gamma_i'(t_i)$ for $i = 1, 2$. On the other hand, since ϕ_i have local extrema at t_i, we have $\phi_i'(t_i) = 0$ for $i = 1, 2$. Now, since $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ are linearly independent vectors in $\mathbb{R}^2$, we can conclude that $\nabla f(\boldsymbol{p}) = \mathbf{0}$.

3. Discriminant test

The discriminant test or the second derivative test is a high point of any exposition of local extrema and saddle points of functions of two real variables. It facilitates easy checking of saddle points in many, but not all, cases. The classical definition of a saddle point given in our introduction is, in fact, tailor-made so that the discriminant test can be proved easily. Some texts (e.g. [2, p. 347]) even take an easier option to *define* a saddle point as a critical point where the discriminant is negative. This may appear a bit like putting the cart before the horse. But the importance of the discriminant test can hardly be overemphasised and it seems imperative that it remains available with our geometric notion of a saddle point.

Let us recall that a *binary quadratic form* (over $\mathbb{R}$) is a polynomial of the form

$$Q(\boldsymbol{h}) = Q(h_1, h_2) := ah_1^2 + 2bh_1h_2 + ch_2^2,$$

where $\boldsymbol{h} = (h_1, h_2)$ is a pair of variables and a, b, c are (real) constants. We say that Q is *positive definite* if $Q(\boldsymbol{u}) > 0$ (similarly, *negative definite* if $Q(\boldsymbol{u}) < 0$) for all $\boldsymbol{u} \in \mathbb{R}^2$, $\boldsymbol{u} \neq \mathbf{0}$. In case Q takes positive as well as negative values, that is, if there are $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ such that $Q(\boldsymbol{u})Q(\boldsymbol{v}) < 0$, then Q is said to be *indefinite*. In this situation, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are necessarily nonzero and they can not be multiples of each other since $Q(t\boldsymbol{h}) = t^2Q(\boldsymbol{h})$ for any $t \in \mathbb{R}$ and $\boldsymbol{h} \in \mathbb{R}^2$.

Let $D \subseteq \mathbb{R}^2$ and $\boldsymbol{p}$ be an interior point of D. Suppose $f\colon D \to \mathbb{R}$ has continuous partial derivatives of first and second order in an open neighbourhood of $\boldsymbol{p}$. Then the *Hessian form* of f at $\boldsymbol{p}$ is the binary quadratic form defined by

$$Q_{\boldsymbol{p}}(\boldsymbol{h}) = Q_{\boldsymbol{p}}(h_1, h_2) := f_{xx}(\boldsymbol{p})h_1^2 + 2f_{xy}(\boldsymbol{p})h_1h_2 + f_{yy}(\boldsymbol{p})h_2^2.$$

With the hypothesis and notation as above, we have the following.

Proposition 2. *If $\nabla f(\boldsymbol{p}) = \mathbf{0}$ and the Hessian form $Q_{\boldsymbol{p}}$ of f at $\boldsymbol{p}$ is indefinite, then f has a strict saddle point at $\boldsymbol{p}$.*

Proof. The basic argument is similar to that used in many texts on calculus, but we provide a sketch for the sake of completeness. Assume that $\nabla f(\boldsymbol{p}) = \mathbf{0}$ and $Q_{\boldsymbol{p}}$ is indefinite. Then there are nonzero $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ such that $Q_{\boldsymbol{p}}(\boldsymbol{u}) < 0$, while $Q_{\boldsymbol{p}}(\boldsymbol{v}) > 0$. By the continuity of the second order partials, there is $\delta > 0$ such that for any $\boldsymbol{q} \in \mathbb{R}^2$ with $\|\boldsymbol{q} - \boldsymbol{p}\| \le \delta$, we have $\boldsymbol{q} \in D$ and $Q_{\boldsymbol{q}}(\boldsymbol{u}) < 0$, while $Q_{\boldsymbol{q}}(\boldsymbol{v}) > 0$. Scaling $\boldsymbol{u}$ and $\boldsymbol{v}$ suitably, we may assume that $\|\boldsymbol{u}\| \le 1$ and $\|\boldsymbol{v}\| \le 1$. Given any $t \in [-\delta, \delta]$ and $\boldsymbol{h} \in \mathbb{R}^2$ with $\|\boldsymbol{h}\| \le 1$, by Taylor's Theorem, there is $\boldsymbol{q} \in \mathbb{R}^2$ on the line joining $\boldsymbol{p}$ and $\boldsymbol{p} + t\boldsymbol{h}$ such that

$$f(\boldsymbol{p} + t\boldsymbol{h}) - f(\boldsymbol{p}) = \nabla f(\boldsymbol{p}) \cdot (t\boldsymbol{h}) + \tfrac{1}{2} Q_{\boldsymbol{q}}(t\boldsymbol{h}) = \frac{t^2}{2}\, Q_{\boldsymbol{q}}(\boldsymbol{h}).$$

Thus, if $\gamma_1, \gamma_2\colon [-\delta, \delta] \to \mathbb{R}^2$ are defined by $\gamma_1(t) := \boldsymbol{p} + t\boldsymbol{u}$ and $\gamma_2(t) := \boldsymbol{p} + t\boldsymbol{v}$, then γ_1 and γ_2 are regular paths intersecting transversally at $\boldsymbol{p}$ such that f has a strict local maximum at $\boldsymbol{p}$ along γ_1 and a strict local minimum at $\boldsymbol{p}$ along γ_2.

Remark 1. The function f as in Example 2(ii) has a strict saddle point at $\mathbf{0}$, but its Hessian form at $\mathbf{0}$, being identically zero, is not indefinite. This shows that the converse of Proposition 2 is not true, in general. We can probe further. Observe that our proof of Proposition 2 actually shows that when the Hessian form is indefinite, the two paths satisfying the requirements for a strict saddle point, can be chosen as straight line segments. We can, therefore, ask if the 'weak converse' is true, that is, if straight line segments suffice to show that a differentiable function has a strict saddle point at $\boldsymbol{p}$, then whether the Hessian form $Q_{\boldsymbol{p}}$ is necessarily indefinite? The following example shows that the answer is negative.

Example 3. [Dog saddle] The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x^3y - xy^3$ has a saddle point at the origin. To see this, it suffices to consider the paths given by $t \mapsto (t, -t/2)$ and $t \mapsto (t, t/2)$. These are straight line segments intersecting transversally at the origin for which the conditions in Definition 1 are satisfied. But the Hessian form of f at the origin is identically zero, and hence not indefinite. The graph of f near the origin is shown in Figure 2 on the right.

In order to apply Proposition 2 to specific examples, it is essential to have a useful characterisation of the Hessian form being indefinite. This is basically a well-known question of linear algebra. (See, e.g. [4, 5].) Again, we include the requisite result and a quick proof for the sake of completeness.

Lemma 1. *Let $Q(\boldsymbol{h}) := ah_1^2 + 2bh_1h_2 + ch_2^2$ be a binary quadratic form. If $ac - b^2 < 0$, then Q is indefinite.*

Proof. Observe that (i) if $a \neq 0$, then $Q(1,0)Q(b,-a) = a^2(ac - b^2) < 0$, (ii) if $a = 0$ and $c \neq 0$, then $Q(0,1)Q(c,-b) = c^2(ac - b^2) < 0$, and (iii) if $a = 0$ and $c = 0$, then $Q(1,1)Q(1,-1) = -4b^2 < 0$. Thus, in any case, Q is indefinite.

In the remainder of this section, let $D \subseteq \mathbb{R}^2$ and $\boldsymbol{p}$ be an interior point of D. Further, let $f\colon D \to \mathbb{R}$ be such that f has continuous partial derivatives of first and second order in an open neighbourhood of $\boldsymbol{p}$. We define the *discriminant* of f at $\boldsymbol{p}$ to be the real number

$$\Delta f(\boldsymbol{p}) := f_{xx}(\boldsymbol{p})f_{yy}(\boldsymbol{p}) - f_{xy}(\boldsymbol{p})^2.$$

With the hypothesis and notation as above, we have the following.

Theorem 1 (Discriminant Test)**.** *If $\nabla f(\boldsymbol{p}) = \boldsymbol{0}$ and $\Delta f(\boldsymbol{p}) < 0$, then f has a strict saddle point at $\boldsymbol{p}$.*

Proof. Apply Lemma 1 to the Hessian form of f at $\boldsymbol{p}$ and use Proposition 2.

In fact, as an application of Proposition 1, we can obtain the following stronger version of Theorem 1.

Corollary 1. *Assume that $\Delta f(\boldsymbol{p}) \neq 0$. Then*

$$f \text{ has a saddle point at } \boldsymbol{p} \iff \nabla f(\boldsymbol{p}) = \boldsymbol{0} \text{ and } \Delta f(\boldsymbol{p}) < 0.$$

In particular, when $\Delta f(\boldsymbol{p})$ is nonzero, f has a saddle point at $\boldsymbol{p}$ if and only if it has a strict saddle point at $\boldsymbol{p}$.

Proof. If f has a saddle point at $\boldsymbol{p}$, then it can not have a strict local extremum at $\boldsymbol{p}$. Hence, by the discriminant test for local extrema of functions of two variables [6, §3.3], $\Delta f(\boldsymbol{p})$ can not be positive. Thus, in view of Proposition 1, we have $\nabla f(\boldsymbol{p}) = \boldsymbol{0}$ and $\Delta f(\boldsymbol{p}) < 0$. The converse follows from Theorem 1. The last assertion follows from the equivalence just proved and Theorem 1.

Remark 2. Example 2(i) can be treated with the help of the discriminant test. On the other hand, if f is as in Example 2(ii) or Example 3, then $\Delta f(\boldsymbol{0}) = 0$, and hence the discriminant test is not applicable. This shows that the converse of Theorem 1 is not true, in general.

4. Examples

The aim of this section is to discuss a variety of examples, which not only illustrate our definition of a saddle point but also enable the reader to compare it with the definition usually found in Calculus texts. In the latter case, we call it a saddle points in the classical sense. Note that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a constant function and $\boldsymbol{p}$ is any point of $\mathbb{R}^2$, then f has a saddle point at $\boldsymbol{p}$ in our sense but not in the classical sense. On the other hand, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) := x^3$, then as we show in Example 4(iii) below, f has a saddle point at the origin in the classical sense, but not in our sense. However, a strict saddle point in our sense is a saddle point in the classical sense. This is the case when the discriminant test (Theorem 1) is applicable.

Example 4. (i) Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := \min\{|x|,|y|\}$ if $xy \geq 0$ and $f(x,y) := -\min\{|x|,|y|\}$ if $xy < 0$. Using the paths γ_1 and γ_2 in Example 1(ii), namely, those given by $t \mapsto (t,-t)$ and $t \mapsto (t,t)$, we see that f has a strict saddle point at $(0,0)$. Thus, a nondifferentiable function can have a saddle point (in our sense).

(ii) Let $c_1, c_2 \in \mathbb{R}$ with $0 < c_1 < c_2$. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := (y - c_1x^2)(y - c_2x^2)$. Using the paths given by $t \mapsto (t, ct^2)$ and $t \mapsto (0,t)$, where $c \in \mathbb{R}$ satisfies $c_1 < c < c_2$, we see that f has a strict saddle point at $(0,0)$. Note that for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we have $f(t, \lambda t) = t^2(\lambda - c_1t)(\lambda - c_2t) > 0$ for $0 < |t| < |\lambda|/c_2$. Also, $f(t,0) = c_1c_2t^4 > 0$ and $f(0,t) = t^2 > 0$ for all $t \neq 0$. Hence f has a strict local minimum at $(0,0)$ along every straight line through the origin. Thus, in this example, straight line segments alone can not work to show that f has a saddle point at $(0,0)$, but a combination of a parabola and a straight line segment does.

(iii) Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := x^m$, where m is an odd positive integer. If $m = 1$, then clearly, f has no saddle points (in any sense). Assume now that $m > 1$. Then f is differentiable and $\nabla f(0,y_0) = (0,0)$ for any $y_0 \in \mathbb{R}$. Since f takes both positive and negative values in every open neighbourhood of $(0,y_0)$, we see that f has a saddle point at $(0,y_0)$ in the classical sense for every $y_0 \in \mathbb{R}$. On the other hand, if it had a saddle point (in our sense) at $(0,y_0)$ for some fixed $y_0 \in \mathbb{R}$, then we would find paths $\gamma_i\colon [a_i,b_i] \to \mathbb{R}^2$ satisfying the conditions of Definition 1. Write $\gamma_i(t) := (x_i(t), y_i(t))$ and let $t_i \in (a_i,b_i)$ be such that $\gamma_i(t_i) = (0,y_0)$. Then 0 is a local extremum of x_i^m at t_i. But since x and x^m have the same sign for any $x \in \mathbb{R}$, it follows that each x_i has a local extremum at t_i. Consequently, $x_i'(t_i) = 0$ for $i = 1,2$, and therefore γ_1 and γ_2 can not intersect transversally at $(0,y_0)$. Thus f does not have any saddle point.

Our next set of examples generalise some of the simplest and most natural types of functions of two variables, such as xy, $x^2 - y^2$, $x^3 - 3xy^2$ and x^m, which we have seen earlier. The arguments in the general case are a bit involved and make good material for starred exercises in calculus texts (although we have yet to see them in print), especially for those who may choose to adopt our definition of saddle point.

Example 5. (i) Let $m, n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers) and $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := x^my^n$. Then

$$f \text{ has a strict saddle point at } (0,0) \iff \text{both } m \text{ and } n \text{ are odd.}$$

The reverse implication is easy. Indeed, if m and n are odd, then $m+n$ is even and $f(t,-t) = -t^{m+n} < 0$, while $f(t,t) = t^{m+n} > 0$ for all $t \neq 0$. Thus it suffices to consider the paths γ_1 and γ_2 in Example 1(ii). To prove the forward implication, first suppose m and n are both even. Then $f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$ and hence f can not have a strict local maximum at $(0,0)$ along any path passing through $(0,0)$. So f can not have a strict saddle point at $(0,0)$. Next, suppose m is odd and n is even. For $i = 1,2$, let $\gamma_i\colon [a_i,b_i] \to \mathbb{R}^2$ be regular paths intersecting transversally at $(0,0)$ such that f has a strict local maximum (or a strict local minimum) along γ_1 (resp: γ_2). Write $\gamma_i(t) := (x_i(t), y_i(t))$ and let $t_i \in (a_i,b_i)$ be such that $\gamma_i(t_i) = (0,0)$. Then there is $\delta_1 > 0$ such that

$$0 < |t - t_1| < \delta_1 \Longrightarrow x_1(t)^m y_1(t)^n < 0 \Longrightarrow y_1(t) \neq 0 \text{ and } x_1(t) < 0,$$

where the last implication follows since n is even and m is odd. Thus, x_1 has a strict local maximum at t_1 and so $x_1'(t_1) = 0$. Similarly, there is $\delta_2 > 0$ such that

$$0 < |t - t_2| < \delta_2 \Longrightarrow x_2(t)^m y_2(t)^n > 0 \Longrightarrow y_2(t) \neq 0 \text{ and } x_2(t) > 0.$$

Consequently, $x_2'(t_2) = 0 = x_1'(t_1)$, which contradicts the assumption that γ_1 and γ_2 intersect transversally at $(0,0)$. The case when m is even and n is odd is similar. Thus, we have shown that if both or one of m and n is even, then f does not have a strict saddle point at $(0,0)$.

(ii) Let $m, n \in \mathbb{N}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := x^m - y^n$. Then

$$f \text{ has a saddle point at } (0,0) \Longleftrightarrow \text{both } m \text{ and } n \text{ are even.}$$

The reverse implication is again easy. Indeed, if m and n are both even, we have $f(0,t) = -t^n < 0$, while $f(t,0) = t^m > 0$ for all $t \neq 0$. Thus it suffices to consider the paths $t \mapsto (0,t)$ and $t \mapsto (t,0)$. To prove the forward implication, observe that if $m = 1$ or $n = 1$, then $\nabla f(0,0) \neq (0,0)$, and hence by Proposition 1, f can not have a saddle point at $(0,0)$. So we now assume that $m \geq 2$ and $n \geq 2$. Suppose m or n is odd. Note that if $g\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x,y) := -f(y,x) = x^n - y^m$, then clearly, f has a saddle point at $(0,0)$ if and only if g does. Thus, we may assume without loss of generality that m is odd. Let us first consider the case when $n \geq m$. Let $\gamma_i\colon [a_i, b_i] \to \mathbb{R}^2$ be paths satisfying the conditions in Definition 1. Write $\gamma_i(t) := (x_i(t), y_i(t))$ and let $t_i \in (a_i, b_i)$ be such that $\gamma_i(t_i) = (0,0)$. Then there is $\delta_1 > 0$ such that

$$|t - t_1| < \delta_1 \Longrightarrow x_1(t)^m \leq y_1(t)^n \Longrightarrow x_1(t) \leq y_1(t)^{n/m},$$

where the last implication follows since m is odd. Since $\gamma_1(t_1) = (0,0)$, we see that $x_1 - y_1^{n/m}$ has a local maximum at t_1, and so $x_1'(t_1) - (n/m) y_1(t_1)^{(n-m)/m} y_1'(t_1) = 0$. It follows that $x_1'(t_1) = y_1'(t_1)$ if $n = m$ and $x_1'(t_1) = 0$ if $n > m$. Similarly, there is $\delta_2 > 0$ such that

$$|t - t_2| < \delta_2 \Longrightarrow x_2(t)^m \geq y_2(t)^n \Longrightarrow x_2(t) \geq y_2(t)^{n/m},$$

and this yields that $x_2'(t_2) = y_2'(t_2)$ if $n = m$ and $x_2'(t_2) = 0$ if $n > m$. Either way, the condition that γ_1 and γ_2 intersect transversally is contradicted. Next, suppose m is odd and $n < m$. In case n is odd, then considering $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) := -f(y,x) = x^n - y^m$, we obtain from the previous case that g, and hence f, does not have a saddle point at $(0,0)$. Thus, let us assume that n is even. Now, as before, there is $\delta_2 > 0$ such that

$$|t - t_2| < \delta_2 \Longrightarrow x_2(t)^m \geq y_2(t)^n \geq 0 \Longrightarrow x_2(t) \geq 0,$$

and this yields $x_2'(t_2) = 0$. Consequently, there is $\xi_2\colon [a_2, b_2] \to \mathbb{R}$ such that $x_2(t) = (t - t_2)\xi_2(t)$ for all $t \in [a_2, b_2]$ and moreover $\xi_2(t) \to 0$ as $t \to 0$. Also, since y_2 is differentiable at t_2, there is $\eta_2\colon [a_2, b_2] \to \mathbb{R}$ such that $y_2(t) = (t - t_2)\,[y_2'(t_2) + \eta_2(t)]$ for all $t \in [a_2, b_2]$ and moreover $\eta_2(t) \to 0$ as $t \to 0$. Thus,

$$|t - t_2| < \delta_2 \Longrightarrow (t - t_2)^m \xi_2(t)^m \geq (t - t_2)^n [y_2'(t_2) + \eta_2(t)]^n,$$

and hence

$$0 < |t - t_2| < \delta_2 \Longrightarrow (t - t_2)^{m-n}\xi_2(t)^m \geq [y_2'(t_2) + \eta_2(t)]^n.$$

Since n is even, upon letting $t \to t_2$, we see that $0 \geq y_2'(t_2)^n = |y_2'(t_2)|^n$, and hence $y_2'(t_2) = 0 = x_2'(t_2)$. So the condition that γ_2 is regular is contradicted.

(iii) Let $m, n \in \mathbb{N}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := x^m + y^n$. Then f never has a saddle point at $(0, 0)$. To see this, note that if m and n are both even, then $f(x, y) > 0$ for all $(x, y) \neq (0, 0)$, and so f cannot have a local maximum along any path passing through the origin. The remaining cases can be proved by arguments similar to those in (ii) above.

(iv) [Generalised Monkey Saddle] Let $n \in \mathbb{N}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := \mathrm{Re}(x + iy)^n$. Note that the surface $z = f(x, y)$ is parametrically given by $x = r\cos\theta$, $y = r\sin\theta$ and $z = r^n \cos n\theta$, where $r \geq 0$ and $-\pi < \theta \leq \pi$. If $n = 1$, then clearly, f has no saddle points. But f has a strict saddle point at $(0, 0)$ if $n \geq 2$. The case when n is even is easy. In this case one easily sees that it suffices to consider the paths given by $t \mapsto (t\cos(\pi/n), t\sin(\pi/n))$ and $t \mapsto (t, 0)$. Next, suppose n is odd and $n > 1$. In this case, f is negative in the sectors

$$\frac{\pi}{2n} < \theta < \frac{\pi}{n} \quad \text{and} \quad -\pi < \theta < -\pi + \frac{\pi}{2n},$$

whereas f is positive in the sectors

$$-\frac{\pi}{2n} < \theta < 0 \quad \text{and} \quad \pi - \frac{\pi}{n} < \theta < \pi - \frac{\pi}{2n}.$$

With this in view, we see that if γ_1 is the parabolic path given by

$$t \longmapsto \left(-t\cos\frac{\pi}{2n} + t^2 \sin\frac{\pi}{2n}, t\sin\frac{\pi}{2n} + t^2\cos\frac{\pi}{2n} \right),$$

then $f(\gamma_1(t)) < 0$ for $t \neq 0$ with $|t|$ small, while if γ_2 is the parabolic path given by

$$t \longmapsto \left(t\cos\frac{\pi}{2n} - t^2 \sin\frac{\pi}{2n}, t\sin\frac{\pi}{2n} + t^2\cos\frac{\pi}{2n} \right),$$

then $f(\gamma_2(t)) > 0$ for $t \neq 0$ with $|t|$ small. Also, γ_1 and γ_2 intersect transversally at $(0, 0)$. Thus, f has a strict saddle point at $(0, 0)$.

(v) Let $n \in \mathbb{N}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) := \mathrm{Im}(x + iy)^n$. Note that the surface $z = g(x, y)$ is parametrically given by $x = r\cos\theta$, $y = r\sin\theta$ and $z = r^n \sin n\theta$, where $r \geq 0$ and $-\pi < \theta \leq \pi$. By arguments similar to those in (iv) above, it can be proved that g does not have a saddle point at $(0, 0)$ if $n = 1$, while g has a strict saddle point at $(0, 0)$ if $n \geq 2$.

Remark 3. Due to the nature of our definition of a saddle point, it is not entirely trivial to show that a specific point is a saddle point of a function. The discriminant test can help in many cases, but if it fails, then one has to painstakingly construct regular paths with the desired properties. To this end, it helps to look at the level curves and to know how the function behaves near the point in question, but still some guessing is needed. Further, when Corollary 1 is not applicable, it becomes even more challenging to show that a point is *not* a saddle point of a given function. For this, we need to logically rule out the existence of regular paths satisfying the properties stated in Definition 1.

Remark 4. As is common in introductory texts on multivariable calculus, we have restricted to functions of two variables. But it is clear that many of the notions and results discussed here extend readily to $\mathbb{R}^n$ in place of $\mathbb{R}^2$. For instance, the notions of paths, regularity, transverse intersections, indefiniteness of a quadratic form, and the Hessian form admit straightforward generalisations. If one uses transversally intersecting regular paths in $\mathbb{R}^n$ to define a saddle point of a function of n variables exactly as in Definition 1, then Proposition 2 continues to hold and Theorem 1 admits an analogue with the condition '$\Delta f(\boldsymbol{p}) < 0$' replaced by '$f_{x_i x_i}(\boldsymbol{p}) f_{x_j x_j}(\boldsymbol{p}) - f_{x_i x_j}(\boldsymbol{p})^2 < 0$ for some i, j with $i \neq j$'. However, the analogue of Proposition 1 for $\mathbb{R}^n$ is not valid if $n > 2$. [Consider, for example, $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x_1, x_2, x_3) := x_1 x_2 + x_3$.] In other words, a saddle point is not automatically a critical point. For this reason, a straightforward analogue of Definition 1 is not very satisfactory. A better option may be to define a real-valued function f on an open subset D of $\mathbb{R}^n$ to have a saddle point at $\boldsymbol{p} \in D$ if there are submanifolds of D whose tangent spaces at $\boldsymbol{p}$ span $\mathbb{R}^n$ such that f has a local maximum at $\boldsymbol{p}$ on one and a local minimum at $\boldsymbol{p}$ on another. Returning to the case $n = 2$, if one wants to consider surfaces more general than those defined by graphs of functions of two variables, another plausible definition for $\boldsymbol{p}$ to be a saddle point of a surface S in $\mathbb{R}^3$ could be that there is a plane P passing through $\boldsymbol{p}$ such that $P \cap S$ is like a 'graph' for which $\boldsymbol{p}$ is a vertex of degree ≥ 4.

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