A different look at Albrecht and White's path counting in grids

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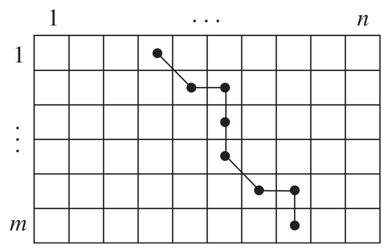
Abstract

In a recent note in this *Gazette*, A.R. Albrecht and K. White considered a problem of counting the total number of paths from a cell in row 1 to a cell in row m of a $m \times n$ grid of cells with restrictions on the moves that are permissible from cell to cell. Albrecht and White determined a triple sum formula for $P_{m,n}$, the number of all such paths. In this brief note, we revisit their problem and utilise generating functions in a natural way to prove an alternative formula — namely, for all $m, n \geq 1$,

$$P_{m,n} = \sum_{r=1}^{m} \binom{m+1+n-r}{n-r} \binom{m-1}{r-1}.$$

In [1] A.R. Albrecht and K. White considered a problem of counting the total number of paths from a cell in row 1 to a cell in row m of a $m \times n$ grid of cells with restrictions on the moves that are permissible from cell to cell. In particular, they allowed moves from cell (i, j) to $(i, j+1)$, $(i+1, j)$, or $(i+1, j+1)$. These correspond to vertical, horizontal, and diagonal moves in the path. An example of such a path is shown in the figure (where we have retained Albrecht and White's notation).



Letting $P_{m,n}$ denote the number of all such paths, Albrecht and White proved a number of facts about $P_{m,n}$ including the following:

- The function $P_{m,n}$ satisfies the recurrence

$$P_{m,n} = 2P_{m,n-1} - P_{m,n-2} + P_{m-1,n} - P_{m-1,n-2} \qquad m \geq 2,\ n \geq 3, \qquad (1)$$

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where

$$P_{1,n} = \frac{n(n+1)}{2}, \qquad P_{m,1} = 1, \qquad P_{m,2} = 2m+1, \qquad m \geq 2.$$

- For all $m, n \geq 1$,

$$P_{m,n} = \sum_{p=1}^{n} \sum_{i=0}^{n-p} \sum_{j=\max\{m-1,n-p-i\}}^{(m-1)+(n-p-i)} \tag{2}$$

$$\frac{j!}{(j-(n-p-i))!\,(j-(m-1))!((m-1)+(n-p-i)-j)!}. \tag{3}$$

Our goal in this note is to prove alternatives for (1) and (2) which follow naturally from the construction of the paths.

We will use two important mathematical facts. First, the binomial theorem states that, for any $n \geq 0$,

$$(x+1)^n = \sum_{j=0}^{n} \binom{n}{j} x^j. \tag{4}$$

We will also use the fact that, for any $n \geq 1$,

$$\frac{1}{(1-x)^n} = \sum_{j=0}^{\infty} \binom{n-1+j}{j} x^j. \tag{5}$$

Identities (4) and (5) can be found in a standard combinatorics text such as [2].

From Table 1 in [1], it appears that $P_{m,n}$ satisfies the following (for any $n \geq 1$):

$$P_{1,n} = n(n+1)/2$$

and

$$P_{m,n} = P_{m-1,n} + 2\sum_{j=1}^{n-1} P_{m-1,j} \tag{6}$$

for $m \geq 2$.

The initial condition $P_{1,n} = n(n+1)/2$ is clear. Next, consider the following constructive way to see the recurrence (6). At the end of each path counted by $P_{m-1,n}$ attach one vertical step. These paths will then be counted by $P_{m,n}$. For any path counted by $P_{m-1,j}$ with $j < n$, complete one of the following two constructions:

- attach one vertical step followed by $n-j$ horizontal steps to the end of the original path, or
- attach one diagonal step followed by $n-j-1$ horizontal steps to the end of the original path.

This construction yields (6).

Now let $G_m(x) = \sum_{n=1}^{\infty} P_{m,n} x^n$ be the generating function of $P_{m,n}$ for a fixed $m \geq 1$. Then (6) can be translated into the language of generating functions as

$$G_m(x) = \frac{x(1+x)^{m-1}}{(1-x)^{m+2}} \tag{7}$$

for all $m \geq 1$. Thanks to (4), (5) and (7), we know that

$$\begin{aligned} G_m(x) &= \sum_{n=1}^{\infty} P_{m,n} x^n \\ &= \frac{x(1+x)^{m-1}}{(1-x)^{m+2}} \\ &= x\left(\binom{m-1}{0} + \binom{m-1}{1}x + \binom{m-1}{2}x^2 + \cdots + \binom{m-1}{m-1}x^{m-1}\right) \\ &\qquad \times \left(\binom{m+1+0}{0} + \binom{m+1+1}{1}x + \binom{m+1+2}{2}x^2 + \cdots\right). \end{aligned}$$

Reading off the coefficient of x^n in this expansion yields the following formula for $P_{m,n}$ for all $m, n \geq 1$:

$$P_{m,n} = \sum_{r=1}^{m} \binom{m+1+n-r}{n-r}\binom{m-1}{r-1} \tag{8}$$

Equation (8) is our alternative to (2).

References

[1] Albrecht, A.R. and White, K. (2008). Counting paths in a grid. *Gaz. Aust. Math. Soc.* **35,** 43–48.

[2] Graham, R.L., Knuth, D.E. and Patashnik, O. (1994). *Concrete Mathematics*, 2nd edn. Addison-Wesley.