A correspondence note on Myerson’s
‘Irrationality via well-ordering’

Scott Duke Kominers∗

I thoroughly enjoyed Myerson’s article [2] on methods of proving irrationality via the well-ordering principle. In this note, I point out a second method by which the well-ordering principle may be used to prove Myerson’s Theorem 2. This approach is a generalisation of MacHale’s recent proof [1] that \( \sqrt{2} \) is irrational.

**Theorem 1.** For \( m \in \mathbb{N} \), \( \sqrt{m} \) is irrational if it is not an integer.

**Proof.** First, we prove the result for squarefree \( m \). We consider the set

\[ \{ a + b \mid a, b \in \mathbb{N}, a^2 = mb^2 \}. \]

By the well-ordering principle for \( \mathbb{N} \), this set has a minimal element \( a_0 + b_0 \). If \( m \) is squarefree, the condition \( a_0^2 = mb_0^2 \) guarantees that \( m \) divides \( a_0 \). Thus, \( a_0 = m\ell \) for some \( \ell \in \mathbb{N} \). But then, \( m^2\ell^2 = a_0^2 = mb_0^2 \), from which it follows that \( m \) divides \( b_0 \). Writing \( b_0 = mr \), we have \( m^3r^2 = m^2\ell^2 \). It follows that \( mr^2 = \ell^2 \).

As \( a_0 + b_0 = m(\ell + r) \), this is a contradiction to the minimality of \( a_0 + b_0 \).

Now, if \( m \) is not squarefree, we may write \( m = m_1^2m_0 \) for \( m_0 \) squarefree. We have \( \sqrt{m_0} \) irrational by the earlier argument, so \( \sqrt{m} = \sqrt{m_1^2m_0} = m_1\sqrt{m_0} \) is irrational, as well.

**References**
