Comment on 'Counting paths in a grid'

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Abstract

In a recent note in the Gazette, Albrecht and White consider $P_{m,n}$, the number of paths from a cell in row 1 to a cell in row m of an $m \times n$ grid of cells. I obtain a simpler recurrence and a simpler closed formula for $P_{m,n}$, and give an asymptotic formula in the case $m = n$ as $m \to \infty$.

This is a comment on the article 'Counting paths in a grid' by A.R. Albrecht and K. White [1]. Let $r_{m,n}$ for all $m, n \geq 1$ be the number of paths a chess-king can take from $(1,1)$ to (m,n), moving up, right, or diagonally up and right. The $r_{m,n}$ form the array

		n					
		1	2	3	4	5	…
m	1	1	1	1	1	1	
	2	1	3	5	7	9	
	3	1	5	13	25	41	
	4	1	7	25	63	129	
	5	1	9	41	129	321	
	⋮						

If we add the columns, we obtain the $a_{m,n}$, where $a_{m,n}$ is the number of paths from $(1,1)$ to (m,q) for some q with $1 \leq q \leq n$. The $a_{m,n}$ form the array

		n					
		1	2	3	4	5	…
m	1	1	2	3	4	5	
	2	1	4	9	16	25	
	3	1	6	19	44	85	
	4	1	8	33	96	225	
	5	1	10	51	180	501	
	⋮						

If again we add the columns, we obtain the $P_{m,n}$, where $P_{m,n}$ is the number of paths from $(1,p)$ to (m,q) for some p,q with $1 \leq p \leq q \leq n$. The $P_{m,n}$ form the

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array

		n					
		1	2	3	4	5	...
m	1	1	3	6	10	15	
	2	1	5	14	30	55	
	3	1	7	26	70	155	
	4	1	9	42	138	363	
	5	1	11	62	242	743	
	⋮						

The $r_{m,n}$, the $a_{m,n}$ and the $P_{m,n}$ **all** satisfy the recurrence

$$x_{m,n} = x_{m,n-1} + x_{m-1,n} + x_{m-1,n-1}.$$

The generating functions are

$$\sum_{m,n\geq 1} r_{m,n} x^m z^n = \frac{xz}{1-x-z-xz},$$
$$\sum_{m,n\geq 1} a_{m,n} x^m z^n = \frac{xz}{(1-z)(1-x-z-xz)},$$
$$\sum_{m,n\geq 1} P_{m,n} x^m z^n = \frac{xz}{(1-z)^2(1-x-z-xz)}.$$

If we write

$$1-x-z-xz = (1-x)(1-z)\left(1-\frac{2xz}{(1-x)(1-z)}\right),$$

we find that

$$r_{m,n} = \sum_{k\geq 0} 2^k \binom{m-1}{k}\binom{n-1}{k},$$
$$a_{m,n} = \sum_{k\geq 0} 2^k \binom{m-1}{k}\binom{n}{k+1},$$
$$P_{m,n} = \sum_{k\geq 0} 2^k \binom{m-1}{k}\binom{n+1}{k+2}.$$

This last formula gives a quick way of calculating $P_{m,n}$.

From these formulas, I can prove that

$$P_{m,m} < \tfrac{1}{2} r_{m+1,m+1} \quad \text{and} \quad P_{m,m}/r_{m+1,m+1} \to \tfrac{1}{2} \quad \text{as } m \to \infty,$$

and so (see [2])

$$P_{m,m} \sim \frac{1}{\sqrt{16\pi\sqrt{2}m}}(\sqrt{2}+1)^{2m+1} \quad \text{as } m \to \infty.$$

We know that

$$P_{m,m} \Big/ \left\{\frac{1}{\sqrt{16\pi\sqrt{2}m}}(\sqrt{2}+1)^{2m+1}\right\} \to 1 \quad \text{as } m \to \infty.$$

We can examine the quantity

$$q = 1 - P_{m,m} \Bigg/ \left\{ \frac{1}{\sqrt{16\pi\sqrt{2}m}} (\sqrt{2}+1)^{2m+1} \right\}.$$

By choosing values of m up to 10^8 and using MAPLE to 40 digits, I find that mq seems to approach a limit as $m \to \infty$, and that limit is $c_1 \approx 0.824524$. If that is indeed the case, then

$$P_{m,m} \sim \frac{1}{\sqrt{16\pi\sqrt{2}m}} (\sqrt{2}+1)^{2m+1} \left(1 - \frac{c_1}{m} + \cdots\right) \quad \text{as } m \to \infty.$$

References

[1] Albrecht, A.R. and White, K. (2008). Counting paths in a grid. *Gaz. Aust. Math. Soc.* **35,** 43–48.

[2] Hirschhorn, M.D. (2000). How many ways can a king cross the board?, *Gaz. Aust. Math. Soc.* **27,** 104–106.