On the enumeration of Pythagorean triples having a fixed base length

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Abstract

An enumeration formula for counting the number of partitions of an integer $n > 1$ having parts in arithmetic progression of common difference two is derived in terms of the number of divisors of n. As a consequence, a formula is obtained giving the number of Pythagorean triples in which n occurs as one of the base lengths in terms of the number of odd and even divisors of n^2.

Introduction

Among the many results of classical number theory attributable to Fermat, one of the most well known asserts that any prime $p \equiv 1 (\bmod 4)$ is always representable as a sum of two squares. This result was later extended by Euler who proved that an integer n is representable as a sum of two squares if and only if when n is expressed as a product of prime powers, every prime factor $p \equiv 3 (\bmod 4)$ occurs with an even exponent. However, it was Jacobi who showed that one could count the number of such representations of n via an enumeration formula, given in terms of positive divisor functions. Indeed, if for each $i \in \{1, 3\}$, one defines $D_i(n)$ as the number of positive divisors d of n such that $d \equiv i (\bmod 4)$, then $r_2(n)$, the number of representations of n as a sum of two squares, is given by

$$r_2(n) = 4(D_1(n) - D_3(n)). \tag{1}$$

(Noting here that the presence of the multiplicative factor of 4 in (1), is to take into account the four possible combinations on the signs of the pair of numbers being squared.) By considering only the sum of squares of positive integers, we see one immediate consequence of the Jacobi formula is that $r_2(n^2)/4 - 1$ counts the number of right-angle triangles with integer side lengths, in which a positive integer n appears as a hypotenuse length. To illustrate take $n = 5$, then as $D_1(n^2) = 3$, $D_3(n^2) = 0$, we have $r_2(n^2) = 12$ and so $r_2(n^2)/4 - 1 = 2$, which corresponds to the two right-angle triangles represented by the triples $(3, 4, 5)$ and $(4, 3, 5)$. Recalling that a Pythagorean triple is a 3-tuple of integers (x, y, z), with $0 < x < y < z$ and satisfying the equation $x^2 + y^2 = z^2$, one may, in view of the previous observation, question whether it is possible to find a similar enumeration formula, in terms of positive divisor functions, for the number of Pythagorean

Received 29 July 2008; accepted for publication 15 June 2009.

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triples in which n appears as one of the base lengths. In this paper, we show that such a formula does indeed exist, and moreover is given by

$$\tfrac{1}{2}(d_0(n^2) + (-1)^{n+1} d_1(n^2) - 1), \tag{2}$$

where, for each $i \in \{0,1\}$, one defines $d_i(n)$ as the number of positive divisors d of n such that $d \equiv i (\mathrm{mod}\, 2)$. It is of interest to note that the expression in (2) can also be written in terms of the simple divisor function $d(n)$, since using the relation $d_1(n) = d(n) - d_0(n)$ and noting $d_0(n) = d(n/2)$ if n is even and 0 otherwise, we find that (2) reduces to the function $f(n)$ where

$$f(n) = \begin{cases} \dfrac{d(n^2) - 1}{2} & \text{for } n \text{ odd,} \\ \dfrac{2d(n^2/2) - d(n^2) - 1}{2} & \text{for } n \text{ even.} \end{cases}$$

The expression in (2) will be derived from a curious enumeration formula which counts the number of partitions of an integer having parts in arithmetic progression of common difference two. We begin in the next section with a small digression on the construction of this partition formula.

Main results

It is well known that the number of consecutive integer partitions of a positive integer n, that is partitions having consecutive integer parts, is one less than the number of odd divisors of n. Indeed, this particular result has rather a long history, as it first appeared as a problem of LeVeque in [3] and has subsequently reappeared as a proven theorem in [2] and [4]. Although there exists a criterion for n to be decomposable into a partition having parts in arithmetic progression with a prescribed common difference (see [1]), it appears that no investigation has been made into the construction of enumeration formulae for such partitions, save for the case of consecutive integer partitions. We now show that at least in the case of partitions with parts in arithmetic progression of common difference two, an enumeration formula, denoted $p_2(n)$, exists and is given in terms of the number of positive divisors $d(n)$ of n.

Lemma 1. *For any integer $n > 1$, the number of partitions of n with parts in arithmetic progression having a common difference of two is given by*

$$p_2(n) = \frac{1}{2}\left(d(n) - 2 + \frac{(-1)^{d(n)+1} + 1}{2}\right). \tag{3}$$

Proof. Recall $a + (a+2) + \cdots + (a + 2(n-1)) = n(n + a - 1)$ and for the partitions in question $a, n \in \mathbb{N}$ with $a \geq 1$ and $n \geq 2$. Consequently, for an integer $N \geq 2$ the value of $p_2(N)$ is equal to the number of representations of the form $N = n(n+k)$, where $n \geq 2$ and $k \geq 0$. Now as $n \geq 2$ and $n + k \geq n$ our task is thus reduced to determining the number of divisors s of N such that $s \neq 1, N$ and $s \leq N/s$. If N is not a perfect square then $d(N)$ is even. Excluding the divisors $s = 1, N$ we see, after grouping the remaining $d(N) - 2$ divisors into pairs of the form $(s, N/s)$, that there must be exactly $(d(N) - 2)/2$ divisors satisfying the above condition. If

$N > 1$ is square then $d(N)$ is odd and $s = \sqrt{N} = N/s$. Thus after excluding the divisors $s \in \{1, \sqrt{N}, N\}$ and again pairing, there must be $(d(N)-3)/2$ divisors s such that $s < N/s$. However, including again $s = \sqrt{N}$, one deduces that there are $(d(N)-1)/2$ divisors $s \le N/s$ such that $s \ne 1, N$. Hence in both cases $p_2(N) = \frac{1}{2}(d(N) - 2 + ((-1)^{d(N)+1} + 1)/2)$.

Remark 1. *If the generating function of $p_2(n)$ is denoted by $f(q)$, that is $f(q) := \sum_{n=2}^{\infty} p_2(n)q^n$, observe that as the coefficient of q^N in the power series expansion of $q^{n^2}/(1-q^n)$ is equal to the number of representations of $N = n(n+k)$, where $k \ge 0$ and $n \ge 2$, then*

$$f(q) = \sum_{n=2}^{\infty} \frac{q^{n^2}}{1-q^n}.$$

In what follows recall that if the 3-tuple of integers (x, y, z) is such that $0 < x < y < z$ and $x^2 + y^2 = z^2$ then, we do not count the 3-tuple (y, x, z) as a Pythagorean triple having either x or y as one of its base lengths. By applying Lemma 1 we can now derive the desired enumeration formula.

Theorem 1. *For any integer $n > 1$ the number of Pythagorean triples in which n appears as one of the base lengths is given by*

$$\tfrac{1}{2}(d_0(n^2) + (-1)^{n+1} d_1(n^2) - 1). \tag{4}$$

Proof. In order to establish (4) our first aim will be to develop an enumeration formula, denoted $s(n)$, which counts the number of representations of an integer $n > 1$ as a difference of squares of two non-negative integers. Once this is achieved we shall deduce (4) from $s(n^2) - 1$, since each representation of the form $n^2 = x^2 - y^2$, having $x, y > 0$, yields a unique Pythagorean triple, where n is one of the base lengths, with the exception of the degenerate case $n^2 - 0^2$.

To begin, we make the following simple observation that the $p_2(n)$ partitions of n must have parts that are either all odd or all even consecutive integers. Denoting the number of partitions having entirely even or odd parts by $\phi(n)$ and $\sigma(n)$ respectively one has $p_2(n) = \phi(n) + \sigma(n)$. Now for $n > 2$ and even, there are exactly $d_1(n/2) - 1 = d_1(n) - 1$ consecutive integer partitions of $\frac{n}{2}$, of the form $\sum_{r=m}^{p} r$ with $p > m$. Consequently, there must be $d_1(n) - 1$ partitions of n of the form $\sum_{r=m}^{p} 2r$ and so $\phi(n) = d_1(n) - 1$. Of course, when n is odd, $\phi(n) = 0$ and so one can set $\phi(n) = (((-1)^n + 1)/2)(d_1(n) - 1)$. Thus from the above decomposition of $p_2(n)$ together with Lemma 1 we find that

$$\begin{aligned}\sigma(n) &= \frac{1}{2}\left(d(n) - 2 + \frac{(-1)^{d(n)+1} + 1}{2}\right) - \frac{(-1)^n + 1}{2}(d_1(n) - 1) \\ &= \frac{1}{2}\left(d_0(n) + (-1)^{n+1} d_1(n) + \frac{(-1)^{d(n)+1} + 1}{2}\right) + \frac{(-1)^n - 1}{2},\end{aligned}$$

where we have used the fact that $d(n) = d_0(n) + d_1(n)$. Recalling that the nth perfect square is equal to the sum of the first n consecutive odd integers, it is clear that each $\sigma(n)$ partition of n must correspond to a unique representation of the form $x^2 - y^2$, where $x, y \in \mathbb{N}$ (the set of non-negative integers). Since by definition each of the $\sigma(n)$ partitions contains at least two summands, we must have $x - y > 1$. However, when $n = 2r + 1$, for some $r \in \mathbb{N}$, one of the $s(n)$ representations must be of the form $n = (r+1)^2 - r^2$, thus $s(n) = \sigma(n) + 1$. If $n = 2r$ then clearly no such difference of consecutive squares representation can exist and so $s(n) = \sigma(n)$. Thus one may set $s(n) = \sigma(n) + (((-1)^{n+1} + 1)/2)$, which for $n > 2$ yields

$$s(n) = \frac{1}{2}\left(d_0(n) + (-1)^{n+1} d_1(n) + \frac{(-1)^{d(n)+1} + 1}{2}\right). \tag{5}$$

However, we also see that (5) holds for $n = 2$ since clearly 2 cannot be expressed as a difference of two squares and formally setting $n = 2$ in (5) gives $s(2) = 0$. Finally, recalling that $d(n^2)$ is always odd, observe from (5) that $s(n^2) - 1$ reduces to the expression in (4), as $n^2 + 1$ and $n + 1$ are of the same parity.

To illustrate Theorem 1 consider firstly the odd integer $n = 21 = 3 \cdot 7$, from which we readily see that $d_0(n^2) = 0$ while $d_1(n^2) = 9$. Thus from (4) there must be exactly four Pythagorean triples having a base length of 21. These are $(21, 28, 35)$, $(20, 21, 29)$, $(21, 72, 75)$ and $(21, 220, 221)$. Secondly, considering the even integer $n = 30 = 2 \cdot 3 \cdot 5$, we readily again see that $d_0(n^2) = 18$ while $d_1(n^2) = 9$. Thus from (4) there must also be exactly four Pythagorean triples having a base length of 30. These are $(30, 72, 78)$, $(16, 30, 34)$, $(30, 40, 50)$ and $(30, 224, 226)$.

Acknowledgment

The author thanks the anonymous referee for the many valuable suggestions made which helped improve the presentation of this paper.

References

[1] Cook, R. and Sharp, D. (1995). Sums of arithmetic progressions. *Fibonacci Quarterly* **33,** 218–221.
[2] Guerin, E.E. (1995). Consecutive integer partitions. *ARS Combinatoria* **39,** 255–260.
[3] LeVeque, W.J. (1965). *Topics in Number Theory.* Addison-Wesley Publishing Company.
[4] Nyblom, M.A. (2001). On the representation of the integers as a difference of non-consecutive triangular numbers. *Fibonacci Quarterly* **39,** 256–263.