On the evaluation of determinants using two order subdeterminants

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Abstract

Some simple algorithms are given for evaluation of determinants using only the second-order subdeterminants together with some illustrative examples.

The traditional methods for hand-calculation of the determinant of an $n \times n$ matrix are based either on simplifying the matrix by performing elementary row or column operations or on expansion by minors along some row or column. A brief overview of the theory of determinants can be found, for example, in [6] and [7]. There are many commonly-used computer packages such as Mathematica or Matlab in which the algorithms to find the determinant of a matrix are based on factorisation in a product of lower and upper matrices. The purpose of this note is to present some methods for evaluation of determinants using only second-order subdeterminants.

Some methods of evaluation of determinant

For a given $n \times n$ matrix, A, fix one of its nonzero element a_{ik} and denote it by α:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,k-1} & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k-1} & a_{2k} & a_{2,k+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i,k-1} & \alpha & a_{i,k+1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,k-1} & a_{nk} & a_{n,k+1} & \cdots & a_{nn} \end{bmatrix}. \tag{1}$$

Proposition 1. *For the determinant of the matrix A the following relation holds*

$$\det(A) = \frac{1}{\alpha^{n-2}} \det(C), \tag{2}$$

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where C is a matrix of the following form:

$$\begin{bmatrix}
\begin{vmatrix} a_{11} & a_{1k} \\ a_{i1} & \alpha \end{vmatrix} & \cdots & \begin{vmatrix} a_{1,k-1} & a_{1k} \\ a_{i,k-1} & \alpha \end{vmatrix} & \begin{vmatrix} a_{1k} & a_{1,k+1} \\ \alpha & a_{i,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1k} & a_{1n} \\ \alpha & a_{in} \end{vmatrix} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\begin{vmatrix} a_{i-1,1} & a_{i-1,k} \\ a_{i1} & \alpha \end{vmatrix} & \cdots & \begin{vmatrix} a_{i-1,k-1} & a_{i-1,k} \\ a_{i,k-1} & \alpha \end{vmatrix} & \begin{vmatrix} a_{i-1,k} & a_{i-1,k+1} \\ \alpha & a_{i,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} a_{i-1,k} & a_{i-1,n} \\ \alpha & a_{in} \end{vmatrix} \\
\begin{vmatrix} a_{i1} & \alpha \\ a_{i+1,1} & a_{i+1,k} \end{vmatrix} & \cdots & \begin{vmatrix} a_{i,k-1} & \alpha \\ a_{i+1,k-1} & a_{i+1,k} \end{vmatrix} & \begin{vmatrix} \alpha & a_{i,k+1} \\ a_{i+1,k} & a_{i+1,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} \alpha & a_{in} \\ a_{i+1,k} & a_{i+1,n} \end{vmatrix} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\begin{vmatrix} a_{i1} & \alpha \\ a_{n1} & a_{nk} \end{vmatrix} & \cdots & \begin{vmatrix} a_{i,k-1} & \alpha \\ a_{n,k-1} & a_{nk} \end{vmatrix} & \begin{vmatrix} \alpha & a_{i,k+1} \\ a_{nk} & a_{n,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} \alpha & a_{in} \\ a_{nk} & a_{nn} \end{vmatrix}
\end{bmatrix}.$$

Proof. Let us perform on columns c_j, $j \neq k$, of the matrix A the following replacements: $c_j \to \alpha.c_j - a_{ij}.c_k$. Then we obtain the following matrix:

$$\begin{bmatrix}
\alpha a_{11} - a_{1k}a_{i1} & \cdots & \alpha a_{1,k-1} - a_{1k}a_{i,k-1} & a_{1k} & \alpha a_{1,k+1} - a_{1k}a_{i,k+1} & \cdots & \alpha a_{1n} - a_{1k}a_{in} \\
\alpha a_{21} - a_{2k}a_{i1} & \cdots & \alpha a_{2,k-1} - a_{2k}a_{i,k-1} & a_{2k} & \alpha a_{2,k+1} - a_{2k}a_{i,k+1} & \cdots & \alpha a_{2n} - a_{2k}a_{in} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha a_{n1} - a_{nk}a_{i1} & \cdots & \alpha a_{n,k-1} - a_{nk}a_{i,k-1} & a_{nk} & \alpha a_{n,k+1} - a_{nk}a_{i,k+1} & \cdots & \alpha a_{nn} - a_{nk}a_{in}
\end{bmatrix}.$$

The kth column is unchanged, in the ith row all elements are 0 except on the kth position. The determinant of the matrix A is then equal to the determinant of this matrix divided by α^{n-1}. Let us expand the determinant of the last matrix by the ith row. We take into account the factor $(-1)^{i+k}$ by multiplying the last $n-i$ rows and the last $n-k$ columns by -1. The determinant of the matrix A is then $1/(\alpha^{n-2})$ times the determinant of the following matrix:

$$\begin{bmatrix}
\alpha a_{11} - a_{1k}a_{i1} & \cdots & \alpha a_{1,k-1} - a_{1k}a_{i,k-1} & a_{1k}a_{i,k+1} - \alpha a_{1,k+1} & \cdots & a_{1k}a_{in} - \alpha a_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha a_{i-1,1} - a_{i-1,k}a_{i1} & \cdots & \alpha a_{i-1,k-1} - a_{i-1,k}a_{i,k-1} & a_{i-1,k}a_{i,k+1} - \alpha a_{i-1,k+1} & \cdots & a_{i-1,k}a_{in} - \alpha a_{i-1,n} \\
a_{i+1,k}a_{i1} - \alpha a_{i+1,1} & \cdots & a_{i+1,k}a_{i,k-1} - \alpha a_{i+1,k-1} & \alpha a_{i+1,k+1} - a_{i+1,k}a_{i,k+1} & \cdots & \alpha a_{i+1,n} - a_{i+1,k}a_{in} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{nk}a_{i1} - \alpha a_{n1} & \cdots & a_{nk}a_{i,k-1} - \alpha a_{n,k-1} & \alpha a_{n,k+1} - a_{nk}a_{i,k+1} & \cdots & \alpha a_{nn} - a_{nk}a_{in}
\end{bmatrix}$$

and this is in fact the matrix C. The proof is complete.

Thus, the determinant of the matrix A of the nth order can be reduced to the evaluation of the determinant of a matrix of order $n-1$. Each element of this reduced matrix is the second-order subdeterminant of the elements in A lying on the intersections of row i and column k with a particular row and column, which always includes the chosen element α. The positions remain the same as in the original matrix without taking care of the signs.By successively applying this formula we come to only one second-order determinant. By hand-calculation, it is practical to choose the element 1 or -1, if any.

As an example, let us evaluate the following determinant from [1]:

$$
\begin{aligned}
\begin{vmatrix}
2 & -1 & 5 & 8 & 3 & -4 \\
0 & 4 & -3 & 4 & 3 & 8 \\
1 & -3 & -2 & 5 & 7 & \mathbf{1} \\
4 & 6 & -4 & 2 & 9 & 0 \\
3 & 5 & -2 & 4 & 7 & 0 \\
2 & 4 & 6 & -3 & 2 & 8
\end{vmatrix}
&=
\begin{vmatrix}
6 & -13 & -3 & 28 & 31 \\
-8 & 28 & 13 & -36 & -53 \\
-4 & -6 & 4 & -2 & -9 \\
-3 & -5 & \mathbf{2} & -4 & -7 \\
6 & -28 & -22 & 43 & 54
\end{vmatrix} \\
&= \frac{1}{2^3}
\begin{vmatrix}
3 & -41 & -44 & -41 \\
23 & 121 & 20 & 15 \\
\mathbf{4} & 8 & -12 & -10 \\
54 & 166 & -2 & -46
\end{vmatrix} \\
&= \frac{1}{2^3 4^2}
\begin{vmatrix}
188 & 140 & 134 \\
-\mathbf{300} & -356 & -290 \\
232 & 640 & 356
\end{vmatrix} \\
&= \frac{-1}{128 \times 300}
\begin{vmatrix}
-24928 & -14320 \\
-109408 & -39520
\end{vmatrix} \\
&= 15145.
\end{aligned}
$$

We denoted in bold the elements used in the next step. In hand-calculation it is always practical to reduce the fractions by common factors in some row or column. Note that if we choose in the above relation $\alpha = a_{11}$ we obtain Chio's formula (see [5]). For the evaluation of the determinant of a nth order matrix, the required multiplication and division operations are $(4n^3 - 3n^2 - 7n + 6)/6$ which is a little less than for the algorithm in [1]. If we add the subtraction operations, the total sum of arithmetic operation to be used is $n^3 - n^2 - n + 1$.

The above formula can be generalised by choosing two or more nonzero elements in some row. For example, let us choose $\alpha = a_{ik} \neq 0$ and $\beta = a_{is} \neq 0$, where $1 \leq k < s \leq n$. Similar to above, we obtain the new formula by replacing the columns of the original matrix A in the following manner:

$$
\begin{aligned}
c_j &\to \alpha \cdot c_j - a_{ij} \cdot c_k \qquad && j = 1, 2, \ldots, s,\ j \neq k, \\
c_j &\to \beta \cdot c_j - a_{ij} \cdot c_s \qquad && j = s+1, \ldots, n.
\end{aligned}
$$

Proposition 2. *For a given matrix A in (1) let us choose $\alpha = a_{ik} \neq 0$ and $\beta = a_{is} \neq 0$. Then the following formula holds:*

$$
\det(A) = \frac{1}{\alpha^{s-2}\beta^{n-s}} \det(C), \tag{3}
$$

where C is the matrix consisting of second-order subdeterminants with a similar form to that in Proposition 1, where the first parameter α is used to form the first $s-1$ columns, while for the remaining columns we use the parameter β.

As an example, let A be a 5×5 matrix with the two chosen nonzero elements $\alpha = a_{32}$ and $\beta = a_{34}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & \alpha & a_{33} & \beta & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Then the following formula holds:

$$\det(A) = \frac{1}{\alpha^{4-2}\beta^{5-4}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & \alpha \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ \alpha & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{14} \\ \alpha & \beta \end{vmatrix} & \begin{vmatrix} a_{14} & a_{15} \\ \beta & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & \alpha \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ \alpha & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{24} \\ \alpha & \beta \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ \beta & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{31} & \alpha \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} \alpha & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} \alpha & \beta \\ a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} \beta & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{31} & \alpha \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} \alpha & a_{33} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} \alpha & \beta \\ a_{52} & a_{54} \end{vmatrix} & \begin{vmatrix} \beta & a_{35} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix}.$$

The above relations can be generalised to

Proposition 3. *Let us choose in the ith row of the matrix (1) the nonzero parameters $a_{i,k_1}, a_{i,k_2}, \ldots, a_{i,k_r}$, where $1 \le k_1 < k_2 < \cdots < k_r < n$, then the following formula holds:*

$$\det(A) = \frac{1}{a_{i,k_1}^{k_2-2} a_{i,k_2}^{k_3-k_2} \ldots a_{i,k_r}^{n-k_r}} \det(C), \tag{4}$$

where C is a matrix of 2×2 determinants formed, as in the previous propositions, progressively using the chosen parameters a_{i,k_j}, skipping to the next parameter at the k_{j+1}th column.

In particular, we can choose all elements of the first row, apart from the first and last: a_{ij}, $j = 2, 3 \ldots, n-1$, if all these elements are nonzero. In this case we obtain the following formula:

$$\det(A) = \frac{1}{a_{i2} a_{i3} \ldots a_{i,n-1}} \det(C), \tag{5}$$

where C is the matrix of 2×2 determinants formed in the above way, skipping to the next chosen parameter with each step.

It is interesting that the position of the first parameter in (3) and (4) has no impact on the above factor. The similar formulas as (3), (4) and (5) hold for the parameters chosen in some column.

As an example let us use the relation (5) to evaluate the following determinant of order n

$$D(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) = \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \ldots & a_1b_1^{n-2} & b_1^{n-1} \\ a_2^{n-1} & a_2^{n-2}b_2 & \ldots & a_2b_2^{n-2} & b_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \ldots & a_nb_n^{n-2} & b_n^{n-1} \end{vmatrix} \tag{6}$$

Firstly, we assume that all parameters are nonzero. Using the identity (5) for the first row, we obtain a determinant of order $n-1$. Reducing all common factors in powers of a_1 and b_1 from all columns of this determinant and factoring out all common factors from each row we obtain

$$\begin{aligned} D(a_1, a_2, \ldots, a_n;\, b_1, b_2, \ldots, b_n) &= (a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)\ldots(a_1b_n - a_nb_1) \\ &\quad \times D(a_2, a_3, \ldots, a_n; b_2, b_3, \ldots, b_n). \end{aligned} \tag{7}$$

Continuing this way we have to evaluate only one second-order determinant

$$D(a_{n-1}, a_n;\, b_{n-1}, b_n) = \begin{vmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n \end{vmatrix} = (a_{n-1}b_n - a_nb_{n-1}).$$

Thus, we obtain the identity

$$D(a_1, a_2, \ldots, a_n;\, b_1, b_2, \ldots, b_n) = \prod_{1\le i<j\le n} (a_ib_j - a_jb_i). \tag{8}$$

This relation also holds in the case that some of the numbers are equal to zero. We may show this for the case $a_1 = 0$. Expanding the determinant (6) by the first row we obtain in this case

$$\begin{aligned} D(a_1, a_2, \ldots, a_n;\, b_1, b_2, \ldots, b_n) &= (-1)^{n+1}b_1^{n-1}a_2a_3\ldots a_n \\ &\quad \times D(a_2, a_3, \ldots, a_n;\, b_2, b_3, \ldots, b_n). \end{aligned}$$

The same relation follows from (7) taking $a_1 = 0$. Note, that the above determinant can be evaluated using the formula for the Vandermonde determinant (see [5]). On the other hand the identity for the Vandermonde determinant follows from (8) taking $b_i = 1$ for all i.

Application to a Hessenberg matrix

The determinants of some Hessenberg matrices were treated by several authors (see e.g. [2], [3], [8]). We shall use relation (2) to compute the determinant of a Hessenberg matrix of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2,n-1} & a_{2n} \\ 0 & a_{32} & a_{33} & \ldots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \ldots & a_{n,n-1} & a_{nn} \end{bmatrix}.$$

Let $a_{11} \neq 0$, then it is easy to see, using one step of the algorithm in Proposition 1, that we have in fact only in the first row the second-order subdeterminants, all remaining rows are only multiplied by factor a_{11}. If we factor this out, we obtain

$$\det(A) = \det \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1,n-1} \\ a_{21} & a_{2,n-1} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ 0 & a_{43} & \cdots & a_{4,n-1} & a_{4n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}. \tag{9}$$

If $a_{11} = 0$ and $a_{21} \neq 0$ we start with this element and we obtain the same relation. If both the terms are equal to 0, the above relation holds trivially. We see that the application of the above relation is very simple. In each step we evaluate only the second-order subdeterminants from the first two rows, the other rows remains the same, and we omit the 0s in the first column. The new matrix is again of the Hessenberg form and we continue this way to come to only one determinant of the second order. A similar relation can be obtained starting from the last lower element.

For example:

$$\begin{vmatrix} 1 & 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 5 & 2 \\ 0 & 3 & 5 & 6 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 & 1 \\ 3 & 5 & 6 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 2 & 4 \\ 0 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 7 & 6 \\ 9 & 8 \end{vmatrix} = 2.$$

In a further example, let $A_{n,x}$ be the $n \times n$ Hessenberg matrix, where $n \geq 3$, in which the elements on the main diagonal are 2s, the elements on the sub- and superdiagonal are 1s, the element in $(1, n)$ is x, for some $x \in \mathbb{C}$, and all others element are 0. For $n = 10$, the determinant position of such a matrix is evaluated in [8] by some algorithm for evaluation of tridiagonal determinats (see also [4]). Let B_n be a tridiagonal matrix obtained if $x = 0$ with the $(1, 2)$ element in the first matrix replaced by 2. For $n = 5$ this matrices are

$$A_{5,x} = \begin{bmatrix} 2 & 1 & 0 & 0 & x \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \qquad B_5 = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

For $a_{n,x} := \det(A_{n,x})$ and $b_n := \det(B_n)$ it is easy to see that:

$$b_n = 2 \quad n \geq 2, \qquad a_{n,x} = n + 1 + (-1)^{n-1} x \quad n \geq 3.$$

Namely, if we apply (9) to b_n we obtain the same form of the order $n - 1$, thus $b_n = b_{n-1}$, and since $b_2 = 2$, we are done for b_n. If we apply (9) to $a_{n,x}$ twice and expand the first row in two summands, we obtain the relation $a_{n,x} = a_{n-2,x} + b_{n-2} = a_{n-2,x} + 2$. Hence for the induction step we have $a_{n+1,x} = a_{n-1,x} + 2 = n + (-1)^{n-2} x + 2 = n + 2 + (-1)^n x$. Since $a_{3,x} = 4 + x$ and $a_{4,x} = 5 - x$, this formula holds for all $n \geq 3$.

In [2] and [3] some Hessenberg matrices are studied with determinants that are Fibonacci numbers. As an example let $H_{n,t}$ be the $n \times n$ Hessenberg matrix in which subdiagonal entries are -1s, the main diagonal entries, except the last one, are 2s, and the entries of each column above the main diagonal alternate between -1s and 1s, starting with -1. The lowest diagonal element is equal to $t+1$, for some $t \in \mathbb{C}$. In [3] the determinant of this matrix was shown to be $h_{n,t} := \det(H_{n,t}) = f_n + t f_{n+1}$, where f_n are Fibonacci numbers. The proof was obtained using a system of two recursive relations for the above determinant and for the determinant of a similar matrix where the $(1,1)$ entry is replaced with 1. In [2] a combinatorial proof was given. If we use the relation (9) to the matrix $H_{n,t}$ and expand the first column in two summands, we obtain directly the relation $h_{n,t} = h_{n-1,t} + h_{n-2,t}$ and then using the properties of Fibonacci numbers we obtain the result by induction.

A special case of a Hessenberg matrix is the tridiagonal matrix

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \dots & 0 & 0 \\ a_2 & b_2 & c_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \dots & a_n & b_n \end{bmatrix}.$$

To find its determinant we can use the above algorithm where in each step we calculate two new elements. This can be organised also as the following recursive calculations:

$$\begin{aligned} d_1 &= b_1, & e_1 &= c_1, & \\ d_r &= b_r d_{r-1} - a_r e_{r-1}, & e_r &= c_r d_{r-1}, & r &= 2,3,\dots,n-1, \\ d_n &= b_n d_{n-1} - a_n e_{n-1}. \end{aligned}$$

The last value is the determinant of our matrix: $\det(A) = d_n$. This recursion can also be written as: $d_r = b_r d_{r-1} - a_r c_{r-1} d_{r-2}$, $r = 3,4,\dots,n$, where $d_1 = b_1$, $d_2 = b_2 d_1 - a_2 c_1$. For example, if we apply this recursion to the above matrix B_n we obtain by induction again $\det(B_n) = 2$, for all n.

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