Book reviews

The Computer as Crucible:
An Introduction to Experimental Mathematics

Jonathan Borwein and Keith Devlin
A.K. Peters, 2008, ISBN-13: 978-1568813431

Whenever a book's preface states its aims, a natural question to ask is whether it succeeds in meeting them. Keith Devlin and Jonathan Borwein, two mathematicians with expertise in different mathematical fields but with a common interest in experimental mathematics, begin this book by saying:

> Our aim in writing this book was to provide a short, readable account of experimental mathematics. It is not intended as a textbook to accompany a course ... In particular, we do not aim for comprehensive coverage of the field; rather, we pick and choose topics and examples to give the reader a good sense of the current state of play in the rapidly growing field of experimental mathematics.

The sleuth-like style and lucid writing certainly make this book an enjoyable read. Many explanations are framed by relevant historical context and tales of mathematicians whose use of experimental mathematics helped them gain insights into difficult problems. Although it was never intended to be a course textbook, it could be used as a supplementary text. Many of the chapters are short, and should be viewed as aperitifs.

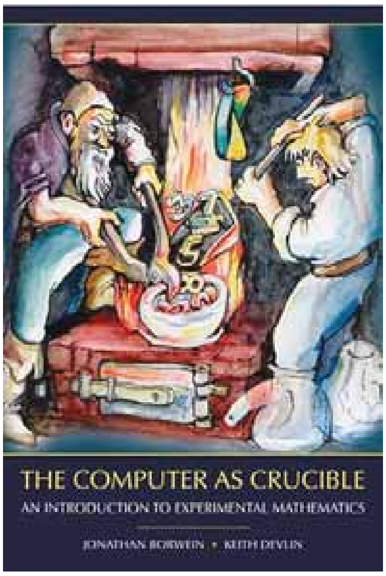
Chapter 1 deals with the important question 'What is experimental mathematics?'. In the authors' own words:

> Experimental mathematics is the use of a computer to run computations — sometimes no more than trial-and-error tests — to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search.

Broadly speaking, it is the use of computers in mathematics as tools in their own right, not simply as numerical calculation aids, '... experimentation is regarded as a significant part of mathematics in its own right ...'.

What kind of experimentation? Here are some of the things described in this book:

- symbolic computation using a computer algebra system such as Maple or Mathematica
- data visualisation
- integer-relation algorithms like PSLQ
- high-precision integer and floating-point arithmetic
- high-precision evaluation of integrals and summation of infinite series
- identification of functions based on their graph characteristics.



It would be very easy to fall into the belief that great mathematicians pluck profound and deep results out of thin air, but some of the mathematical greats (Gauss, Euler, Fermet, Riemann, . . .) were confirmed experimenters who would spend many hours carrying out calculations in order to discover new mathematical avenues worth pursuing. The 72-year-old Gauss recounted in a letter to the astronomer, Johann Encke, that, as a young boy of 15, armed with a table of logarithms he 'frequently spent an idle quarter of an hour to count another chiliad here and there' [1], which led to his estimate of the density of prime numbers; "... Gauss was very clearly an 'experimental mathematician' of the first order."

Chapter 2 gives a brief introduction to the PSLQ algorithm, an integer relation algorithm developed by Helaman Ferguson. Given any real coefficients $\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_n$ and a precision ε, an integer relation algorithm uses high-precision arithmetic to find integer coefficients $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_0 \neq 0$ and

$$|\lambda_0 \mathrm{a}_0 + \lambda_1 \mathrm{a}_1 + \cdots + \lambda_n \mathrm{a}_n| < \varepsilon$$

or else it tells you such expression exists within a ball of a given radius about the origin.

Chapter 3 ('What Is That Number?') introduces inverse symbolic calculators as tools to recognise numbers, and combined with Sloane's online *Encyclopedia of Integer Sequences*, describes a technique for determining closed forms of sequences.

I have more than a passing interest in Riemann's zeta function, the topic of Chapter 4 ('The Most Important Function in Mathematics'); I found it interesting though perhaps a little short. I particularly liked the quote about British soccer player, Wayne Rooney, contrasting him with David Beckham: 'There is more chance of him [Rooney] proving Riemann's Hypothesis than wearing a sarong'!

I'm certain physicists will find Chapter 5 ('Evaluate the Following Integral') interesting, especially given the authors' collaborations in computing closed forms of definite integrals arising in physics.

Chapter 9 ('Take It to the Limit') contains three worked examples of finding closed forms for infinite sums, and Chapter 10 ('Danger! Always Exercise Caution When Using the Computer') contains sobering stories and examples of some of the pitfalls faced by experimental mathematicians. Here is one:

$$\operatorname{sinc}(x) = \frac{\sin x}{x},$$

$$I_1 = \int_0^\infty \operatorname{sinc}(x)\,\mathrm{d}x,$$

$$I_2 = \int_0^\infty \operatorname{sinc}\left(\tfrac{x}{3}\right)\mathrm{d}x,$$

$$I_3 = \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\tfrac{x}{3}\right)\operatorname{sinc}\left(\tfrac{x}{5}\right)\mathrm{d}x, \quad \ldots$$

$$I_8 = \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\tfrac{x}{3}\right)\operatorname{sinc}\left(\tfrac{x}{5}\right)\ldots\operatorname{sinc}\left(\tfrac{x}{15}\right)\mathrm{d}x.$$

A computer algebra system (CAS) will discover that $I_1 = I_2 = I_3 = \cdots = I_7 = \pi/2$ but $I_8 = 0.499999999992646\pi$.

On finding this, the authors suspected a bug in the CAS software. But there is no bug!

The book provides tantalising examples and suggestions to whet the reader's appetite in the form of an 'Explorations' section at the end of each chapter. These are not exactly exercises but there is a corresponding 'Answers and Reflections' chapter at the end of the book. Interested readers will find many of these topics expanded upon in [2].

I thoroughly enjoyed reading this short introduction to experimental mathematics. It will no doubt appeal to a broad mathematical audience, both professional and amateur alike. If I have one complaint (well, more of a request), it would be a much longer chapter on evaluating definite integrals! But then, in the words of G.H. Hardy, 'I could never resist a definite integral' [3].

References

[1] Havil, J. (2003). *Gamma: Exploring Euler's Constant.* Princeton University Press, New Jersey.
[2] Bailey, D. *et al.* (2007). *Experimental Mathematics in Action.* A.K. Peters, MA.
[3] Macdonald, H.S.M. (1999). *The Beauty of Geometry: Twelve Essays.* Dover Publications, New York.

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Thinking About Gödel and Turing: Essays on Complexity, 1970–2007

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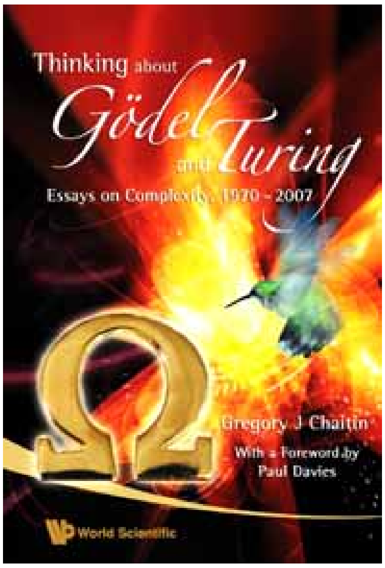
Gregory Chaitin's *Thinking About Goedel and Turing: Essays on Complexity, 1970–2007* contains 23 papers, all by Chaitin, on a wide range of topics. Although the theme of complexity is common to all the papers, many other related issues are also discussed. Chaitin has things to say about Gödel's theorem, about randomness in arithmetic, the foundations of mathematics, the role of complexity and simplicity in scientific method and their connection to the notion of explanation, the similarities between mathematics and physics and his own 'digital philosophy'.

Perhaps Chaitin's best-known contribution to mathematics is his demonstration of the existence of the so-called 'Ω' number, which gives the probability that any randomly chosen computer programme will eventually halt. Chaitin has shown that the binary expansion of Ω is random in the sense that there can be no rule for generating this sequence of '1's and '0's that is shorter than the sequence itself.



Chaitin goes on to draw a number of broadly 'philosophical' conclusions from this. In particular, Chaitin feels that mathematicians have not taken Gödel's Incompleteness Theorems sufficiently seriously. He says that mathematicians tend to 'carry on' with their own mathematical work as before. But for Chaitin, this is a mistake. He argues that 'incompleteness ... [in mathematics] is natural and pervasive'. He sees certain aspects of the history of mathematics as supporting this idea. In particular, he points out (p. 72) that 'many mathematical problems have remained un-solved for hundreds and even thousands of years' and that the reason for this might not lie within mathematicians themselves, but might rather be due to 'the incompleteness of their axioms'. He goes on to suggest that mathematicians should therefore adopt a more liberal attitude to what is to count as an axiom. (He suggests 'There are no odd perfect numbers' as a possible candidate. He also mentions the Riemann hypothesis.)

These ideas lead Chaitin to develop a view of mathematics according to which the subject resembles empirical sciences more closely than is generally thought. In the empirical sciences, laws are not accepted because they are 'self-evident' or '(epistemically) necessary' or 'analytically true'; rather, they are accepted because they explain a lot of other things. Roughly, they are accepted because a lot of other statements we know to be true follow from them. Chaitain suggests that mathematicians ought to view axioms in a way similar to the way scientists view laws: they are to be accepted if we know they have a lot of true consequences. Chaitin acknowledges, however, that most mathematicians would see this suggestion as 'ridiculous'.

Another recurring theme in the book is simplicity and complexity in science. Chaitin notes that since the Ancient Greeks, scientists have believed nature is simple. He produces a number of very interesting quotes, from Plato through to more recent figures such as Weyl, Einstein and Feynman who all agree that it is very important that our theories of the physical world be as simple as possible. He pays special attention to Leibniz, and acknowledges him as a precusor of some of his own ideas.

Chaitin asks: 'What, precisely, is simplicity?' and 'Why is it so desirable?' Briefly, his answer to the first question is that simplicity is compressibility; his answer to the second is that compressibility gives comprehension. He writes: 'a scientific theory is a computer program, and the smaller, the more concise the program, the better the theory' (p. 211).

Chaitin notes that this view is based on a number of assumptions. One of the assumptions is that 'the choice of computer or programming language is not too important'. He goes on to note that 'This is debatable' (p. 212). However, I believe that there is a serious difficulty here that Chaitin has not sufficiently addressed. Empirical theories of the physical or material world will contain non-mathematical terms referring to classes or kinds of things. For example, the simple 'theory', 'All crows are black', contains the terms 'crow' and 'black'. Different languages will contain different terms. Are all *possible* languages to be permitted in expressing theories of the physical world? It is easy to see that if all possible languages are permitted, then simplicity becomes quite useless as a way discriminating between good theories and bad. Let us introduce a language that contains a term 'runcible'. It is possible to define 'runcible', but the definition is very complex. Let O, O*, O** and so on be all the material objects. (We do need to assume the number of material objects is countable.) Let P be the properties of O, P* be the properties of O* and so on. Then we can say X is runcible if and only if X is O and has P, or X is O* and has P*, and so on. Then 'All material objects are runcible' will tell us the truth about every material object. There is, moreover, a sense in which it is a very simple theory: it can be stated in five words of English. But it is a lousy theory.

Intuitively, what is wrong with 'All material objects are runcible' is that it is very complex, but the complexity is contained in the definition of runcible. We might therefore suggest that in evaluating the complexity of a theory, we must first translate it to genuine or legitimate words such as 'crow' or 'black'. But this raises a new question: "What is to count as a 'genuine' or 'legitimate' term?" This has proved enormously difficult to answer! This is not the place to review these difficulties, but some of the difficulties that have most exercised philosophers, at least, are discussed in Nelson Goodman's *Fact, Fiction and Forecast* (Harvard University Press, 1955, especially Chapter 3).

There are very many other ideas dealt with in Chaitin's book. The author's enthusiasm for mathematics, for ideas generally, and for life, comes through clearly. The writing style is informal, and much of it reads like transcripts of lectures and addresses. This book is perhaps not everyone's cup of tea, but for anyone who wants a book bubbling with ideas, written by an author with a passion for his subject, I would strongly recommend it.

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