Maximum number of vertex-disjoint complete subgraphs

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Abstract

In this paper, we will show that the maximum number of disjoint complete subgraphs K_k in a complete multipartite graph $K_{m_1,m_2,\ldots,m_r}$ with $m_1 \geq m_2 \geq \cdots \geq m_r$ is

$$\min_{1\leq i\leq k} \left\lfloor \frac{m_i + m_{i+1} + \cdots + m_r}{k+1-i} \right\rfloor.$$

Introduction

A graph is multipartite if the set of vertices in the graph can be divided into non-empty subsets, called parts, such that no two vertices in the same part have an edge connecting them. Furthermore, a complete multipartite graph is a multipartite graph such that any two vertices that are not in the same part have an edge connecting them. We denote a complete multipartite graph with r parts by $K_{m_1,m_2,\ldots,m_r}$ where m_i is the number of vertices in the ith part of the graph. For convenience, we arrange parts such that $m_1 \geq m_2 \geq \cdots \geq m_r$. In [1], Sitton discussed the question 'How many edges can there be in a maximum matching in a complete multipartite graph?'. Sitton proved that

Theorem 1 ([1]). *Given any complete multipartite graph $K_{m_1,m_2,\ldots,m_r}$, with $m_1 \geq m_2 \geq \cdots \geq m_r$, the size of a maximum matching is*

$$M = \min\left\{ \sum_{i=2}^{r} m_i, \left\lfloor \tfrac{1}{2}\sum_{i=1}^{r} m_i \right\rfloor \right\},$$

where $\lfloor x \rfloor$ is the integer part of x.

Note that the size of a maximal matching is the maximum number of vertex-disjoint complete subgraphs K_2 in a complete multipartite graph $K_{m_1,m_2,\ldots,m_r}$. A natural generalisation is to find the maximum number of vertex-disjoint complete subgraphs K_k in a complete multipartite graph $K_{m_1,m_2,\ldots,m_r}$ for any value of k. The main result of this note is the following theorem.

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Theorem 2. *Let $k \leq r$ be positive integers and $m_1 \geq \cdots \geq m_r$ be r positive integers. For any complete multipartite graph $K_{m_1,\ldots,m_r}$, let M be the maximum number of vertex-disjoint complete subgraphs K_k. Then*

$$M = \min_{1\leq i\leq k} \left\lfloor \frac{m_i + m_{i+1} + \cdots + m_r}{k+1-i} \right\rfloor.$$

Some lemmas

Before proving Theorem 2, we need the following lemmas.

Lemma 1. *Let M be the maximum number of vertex-disjoint complete subgraphs K_k in a complete multipartite graph $K_{m_1,\ldots,m_r}$. Then*

$$M \leq \min_{1\leq i\leq k} \left\lfloor \frac{m_i + m_{i+1} + \cdots + m_r}{k+1-i} \right\rfloor.$$

Proof. Fix i such that $1 \leq i \leq k$. In any complete subgraph K_k of $K_{m_1,\ldots,m_r}$, there is at most one vertex of this subgraph in each part. Thus, there are at most $i-1$ vertices of this subgraph in $i-1$ parts of sizes $m_1, \ldots, m_{i-1}$. So there are at least $k+1-i$ vertices of this subgraph in $r+1-i$ parts of sizes $m_i, \ldots, m_r$. Since M is the maximum number of vertex-disjoint complete subgraphs K_k in $K_{m_1,\ldots,m_r}$, $M(k+1-i) \leq m_i + \cdots + m_r$. The lemma follows.

Lemma 2. *Suppose that $k \geq 2$. Let M be the maximum number of vertex-disjoint complete subgraphs K_k in a complete multipartite graph $K_{m_1,\ldots,m_r}$ with $m_1 \geq \cdots \geq m_r \geq 1$. If*

$$\frac{m_1 + \cdots + m_r}{k} = \min_{1\leq i\leq k} \frac{m_i + m_{i+1} + \cdots + m_r}{k+1-i}$$

then

$$M \geq \left\lfloor \frac{m_1 + \cdots + m_r}{k} \right\rfloor.$$

Proof. Set $T = \lfloor (m_1 + \cdots + m_r)/k \rfloor$. We will construct T vertex-disjoint complete subgraphs K_k of $K_{m_1,\ldots,m_r}$. The idea is to arrange all but at most $k-1$ vertices of $K_{m_1,\ldots m_r}$ into a $k \times T$ array such that all elements in any column are distinct. We have

$$\frac{m_1 + \cdots + m_r}{k} \leq \frac{m_2 + \cdots + m_r}{k-1},$$

which implies that $(k-1)m_1 \leq m_2 + \cdots + m_r$ or $m_1 \leq T$. Our algorithm is as follows. We fill in entries of a $k \times T$ array eventually by vertices of the 1st, 2nd, $\ldots$, rth parts (the ith part has m_i vertices) from top to bottom and left to right. We have $m_i \leq m_1 \leq T$ so there do not exist two vertices which are both in the same part and the same column. Therefore, each column gives us a complete subgraph of order k. This concludes the proof of the lemma.

Proof of Theorem 2

We will prove this theorem by induction on k. If $k = 1$ then the number of complete subgraph K_1 (a single vertex) in a complete multipartite graph $K_{m_1,\ldots,m_r}$

is $M = m_1 + \cdots + m_r$. Hence the statement holds for $k = 1$. Now suppose the statement holds for all positive integers less than k. We will show that the statement also holds for k. Note that we can assume $k \leq r$, otherwise there is no K_k complete subgraph of $K_{m_1,\ldots,m_r}$. Suppose that

$$T = \frac{m_j + \cdots + m_r}{k+1-j} = \min_{1 \leq i \leq k} \frac{m_i + m_{i+1} + \cdots + m_r}{k+1-i}$$

for some $1 \leq j \leq k$. We have two separate cases.

1. Suppose that $j = 1$. Then from Lemma 2, $M \geq T$. But from Lemma 1, $M \leq T$. Hence $M = T$.
2. Suppose that $j > 1$. Then

$$T = \frac{m_j + \cdots + m_r}{k+1-j} \leq \frac{m_{j-1} + \cdots + m_r}{k+1-(j-1)},$$

which implies that

$$T = \frac{m_j + \cdots + m_r}{k+1-j} \leq m_{j-1}.$$

Hence $T \leq m_i$ for all $i \geq j-1$. From Lemma 2, we can construct T vertex-disjoint complete subgraphs of order $k+1-j$ in $K_{m_j,\ldots,m_r}$. Since $m_i \geq T$ for all $i \geq j-1$, for each complete subgraph of order $k+1-j$, we can add one vertex in the part of size m_i for each $i = 1, \ldots, j-1$ in order to make this subgraph of order k. It is clearly that we obtained T vertex-disjoint complete subgraphs of order k. Thus $M \geq T$. But from Lemma 1, $M \leq T$. Hence $M = T$.

This concludes the proof of the theorem.

References

[1] Sitton, D. (1996). Maximum matching in complete multipartite graphs. *Electronic Journal of Undergraduate Mathematics* **00,** 6–16.