Puzzle corner

Norman Do*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 9 is 1 November 2008. The solutions to Puzzle Corner 9 will appear in Puzzle Corner 11 in the March 2009 issue of the *Gazette*.

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Lucky lottery

Fifty players take part in a lottery in which they must write down the numbers from 1 up to 50 in some order. Fifty balls, numbered from 1 up to 50, are drawn one by one from a barrel to provide the winning sequence. The players compare this to their own sequences and earn one dollar for each number which matches. Furthermore, the jackpot is awarded to any player who is lucky enough to have the winning sequence itself. Given that each player wins a different amount of money, prove that at least one of them must have won the jackpot.

Ultramagic square

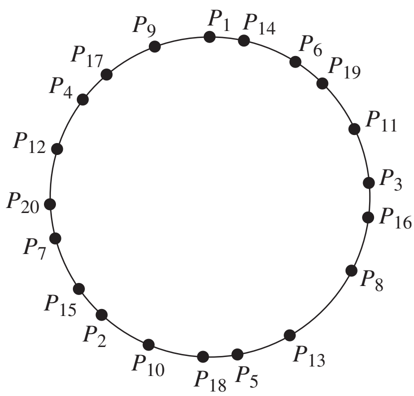
A 9×9 grid is filled with the numbers from 1 up to 81. If the product of the numbers in the kth row is equal to the product of the numbers in the kth column for all k, then we say that the square is *ultramagic*. Does there exist an ultramagic square?

*Department of Mathematics and Statistics, The University of Melbourne, VIC 3010.
E-mail: N.Do@ms.unimelb.edu.au

Cakes and boxes

A triangular cake and a triangular box are congruent, but are mirror images of each other. We would like to cut the cake into two pieces which can fit together in the box without turning either piece over.

(a) Show that this is possible if one angle of the triangle is three times as large as another.
(b) Show that this is possible if one angle of the triangle is obtuse and twice as large as another.



Golden circle

Let P_1 be a point on a circle whose circumference is equal to the golden ratio $\phi = (1+\sqrt{5})/2$. Let P_2 be the point on the circle which is one unit of arc length along from P_1 in the clockwise direction. Let P_3 be the point on the circle which is one unit of arc length along from P_2 in the clockwise direction, and so on. Suppose that you mark the points $P_1, P_2, \ldots, P_n$ for some positive integer n. Prove that if P_i and P_j are adjacent marked points on the circle, then $|i-j|$ is a Fibonacci number.

Robots in mazes

A maze is an 8×8 chessboard with walls along the four sides and between some pairs of adjacent squares. A robot trapped in the maze can walk from one square to an adjacent one as long as there is no wall between them. If the robot can visit every square on the chessboard from some initial position, then the maze is called *good* and otherwise, it is called *bad*.

1. Are there more good mazes or bad mazes?

A *proper* maze is a good maze which has one square marked START and another one marked FINISH. A *program* for the robot is a finite sequence of moves: UP, DOWN, LEFT or RIGHT. A robot will move to the adjacent square in the given direction unless there is a wall blocking it, in which case, it remains on the same square.

2. Suppose that the robot begins on the square marked START. Does there exist a program which will eventually lead the robot to the square marked FINISH, no matter which proper maze is given?

Solutions to Puzzle Corner 7

The \$50 book voucher for the best submission to Puzzle Corner 7 is awarded to Jamie Simpson.

Physicists and chemists

Solution by Warren Brisley: Each mathematician has a liar to their left, so there are at least N liars. However, each physicist has a truth-teller to their left, so there are at least N truth-tellers. There are $2N$ people seated around the table, so there must be precisely N liars and N truth-tellers. Since the number of liars among the mathematicians and the number of liars among the physicists are equal, N must be even.

Sums of alternating sums

Solution by Tony Watts: Consider any subset $X \subseteq \{1, 2, \ldots, n\}$ which does not include n as an element and suppose that its alternating sum is s. Now consider the subset $X' = X \cup \{n\}$ with alternating sum s'. Note that, in the expressions for the alternating sums of X and X', all numbers apart from n will appear with opposite signs. Since n appears with a positive sign in the expression for the alternating sum of X', it follows that $s + s' = n$.

The 2^n subsets of $\{1, 2, \ldots, n\}$ consist of 2^{n-1} which do not include n as an element and 2^{n-1} which do include n as an element. Furthermore, they form 2^{n-1} pairs (X, X') whose sum of alternating sums is precisely n. Hence, the sum of the alternating sums over all subsets of $\{1, 2, \ldots, n\}$ is $n2^{n-1}$.

Rational or irrational?

Solution by David Angell: If A_k was rational, then it would have an eventually periodic decimal expansion. But it is clear that A_k contains arbitrarily long strings of zeros, because 10^{nk} appears in the decimal expansion of A_k for every positive integer n. So the decimal expansion would consist entirely of zeros from some point onwards. Since this is not the case, there is no value of k for which A_k is rational.

David also points out that this is a special case of the following theorem of Kurt Mahler:

> *Let f be a non-constant polynomial with rational coefficients such that $f(k)$ is a positive integer for all positive integers k. Let A be the real number between 0 and 1 formed by writing $f(1), f(2), f(3), \ldots$ in order after the decimal point. Then A is transcendental and not a Liouville number.*

Polygons and rectangles

Solution by Dave Johnson: Given a convex polygon $\mathcal{P}$ of area 1, let A and C be points of $\mathcal{P}$ whose distance is maximal. Let a and c be the lines through A and

C, respectively, perpendicular to AC. Now let B and D be points of $\mathcal{P}$, one on each side of AC and at maximal distance from it. Let b and d be the lines through B and D, respectively, parallel to AC. Denote the quadrilateral $ABCD$ by $\mathcal{Q}$ and the rectangle bounded by the lines a, b, c, d by $\mathcal{R}$. The maximality conditions and the convexity of $\mathcal{P}$ ensure that $\mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{R}$ and, by construction, we have $\text{area}(\mathcal{Q}) = \frac{1}{2}\text{area}(\mathcal{R})$. Finally, we conclude that

$$\text{area}(\mathcal{R}) = 2 \times \text{area}(\mathcal{Q}) \leq 2 \times \text{area}(\mathcal{P}) = 2.$$

We note that this proof works for any plane convex set $\mathcal{P}$ of area 1. Furthermore, the constant 2 cannot be replaced by a smaller number, as evidenced by taking $\mathcal{P}$ to be the isosceles right-angled triangle with hypotenuse of length 2.

The broken calculator

Solution by Jamie Simpson: We will prove the stronger result that every number of the form $\sqrt{p/q}$, where p and q are relatively prime positive integers, can be produced on the calculator. The proof will proceed by induction on $p+q$.

Since $\cos 0 = \sqrt{1/1}$, the statement holds when $p + q = 2$. Now suppose that it holds for all $p + q < n$ and consider $\sqrt{a/b}$ where $a + b = n$. Note that

$$\tan \circ \cos^{-1} \circ \sin \circ \tan^{-1} \sqrt{a/b} = \tan \circ \cos^{-1} \sqrt{a/(a+b)} = \sqrt{b/a}.$$

So we may assume, without loss of generality, that $b > a$. By the induction hypothesis, we can produce $\sqrt{a/(b-a)}$ on the calculator so we can also produce the number

$$\sin \circ \tan^{-1} \sqrt{a/(b-a)} = \sqrt{a/b}.$$

The desired result now follows by induction.

Chessboard puzzles

Solution based on work submitted by Gerry Myerson:

(1) First, we prove the following lemma: if $ABCD$ is any square and P a point in the plane of the square, then $PA^2 + PC^2 = PB^2 + PD^2$. After translation, rotation and dilation, we can assume that $ABCD$ is the unit square in the first quadrant of the Cartesian plane. So if $P = (x, y)$, then one side of the equation is $[x^2 + y^2] + [(x-1)^2 + (y-1)^2]$ and the other is $[x^2 + (y-1)^2] + [(x-1)^2 + y^2]$, both of which are equal.
Now consider the chessboard divided into sixteen 2×2 blocks in the natural way. Such a block consists of four unit squares, with two diagonally opposite ones black and the other two white. Therefore, the result follows by applying the above lemma on the centres of these four squares and summing up over the sixteen 2×2 blocks.

(2) The smallest possible score is 9 which can be achieved by labelling the squares 1 to 8 across the top row, 9 to 16 across the second row, and so on, down to 57 to 64 along the bottom row. To see that it is impossible to do better, note that one can travel from the square labelled 1 to the square labelled

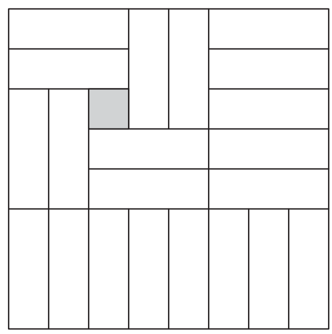
64 in at most 7 king moves. In particular, there is a path from one to the other which passes through two squares which share a common side or vertex whose labels have a difference of at least $(64 - 1)/7 = 9$.

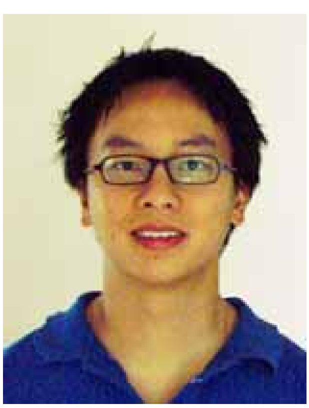
(3) Consider labelling the squares of the chessboard in the following two ways.

1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3

2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2

In either case, the placement of a 3×1 rectangle will cover one square of each label. In both diagrams, there are 21 squares labelled 1, 22 squares labelled 2 and 21 squares labelled 3. It follows that the 1×1 square must be placed on one of the squares which is labelled 2 in both diagrams. Therefore, the 1×1 square must be placed on one of the four shaded squares. In this case, the tiling is given by the following diagram and its rotations.





Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.