Osculation by circumcircles of a pantograph

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Abstract

The circumcircles of the cyclic quadrilaterals in the family $Q(\Lambda)$ of all quadrilaterals for which Λ is the median parallelogram osculate an octic curve, which is described. When Λ is a rhombus this curve reduces to a pair of double ellipses.

In a previous paper [1] we showed that, given a parallelogram $\Lambda = \mathrm{FGHI}$, there exists a doubly infinite family $\mathcal{Q}(\Lambda)$ of quadilaterals, namely all quadrilaterals having three (and therefore all four) of the vertices of Λ as the midpoints of its sides, taken in order; and within $\mathcal{Q}(\Lambda)$ there exists a singly infinite family of cyclic quadrilaterals, whose centres have as locus a rectangular hyperbola $\mathrm{Cen}(\Lambda)$ through the vertices of Λ (or an orthogonal line-pair, namely the diagonal lines, when Λ is a rhombus). We now wish to describe the curve Γ which these circumcircles osculate. Let $\mathcal{C}(\Lambda)$ denote the family of circumcircles.

Γ for a rhombus

We start with the easier case, when Λ is a rhombus. Write f for the side length, ω for the angle, take $0 < \omega \leq \frac{1}{2}\pi$ and $s = \sin\frac{1}{2}\omega$, $c = \cos\frac{1}{2}\omega$, O for the centre of Λ, and take cartesian axes Ox, Oy with Ox parallel to and in the direction FG[1]. The diagonal lines are $\mathrm{s}x - \mathrm{c}y = 0$, $\mathrm{c}x + \mathrm{s}y = 0$. Let $\Theta(u)$ denote the circle of the family $\mathcal{C}(\Lambda)$ having centre $\mathrm{U}(u) = (\mathrm{c}u, \mathrm{s}u)$, $u = |\mathrm{OU}|$, which lies on the longer diagonal IG. See Figure 1.

The equation of $\Theta(u)$ is $\theta(u) = 0$ where (see [1, Theorem 5])

$$\theta(u) := x^2 + y^2 - 2(\mathrm{c}x + \mathrm{s}y)u - f^2 - \mathrm{c}^2\mathrm{s}^{-2}u^2\,. \tag{1}$$

To find the osculated curve we look for the limiting positions of the intersections of circles $\Theta(u)$, $\Theta(u_1)$ as $u \to u_1$. From the equations of these two circles we find

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[1] The notation of [1] is here amended as follows: symbols S, C, E, X, ξ, η are replaced by s, c, e, O, x, y respectively.

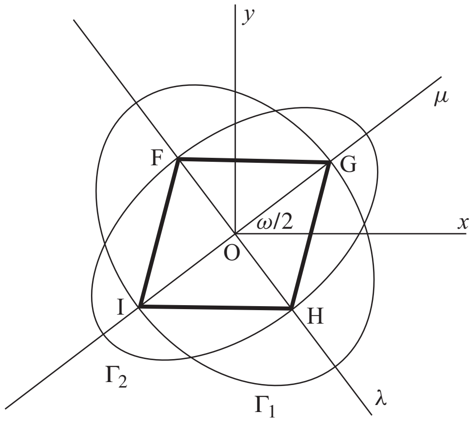

Figure 1. Rhombus Λ and the two ellipses Γ_1, Γ_2.

by subtraction

$$\mathrm{c}x + \mathrm{s}y = -\mathrm{c}^2(2s)^{-1}(u + u_1)\,; \tag{2}$$

then eliminating x from this and $\theta(u_1) = 0$ gives

$$\mathrm{s}^4y^2 + \mathrm{s}^3\mathrm{c}^3(u + u_1)y + 4^{-1}\mathrm{c}^4(u^2 + u_1^2 + 2(1 + 2\mathrm{s}^2)uu_1) - \mathrm{s}^4\mathrm{c}^2f^2 = 0\,.$$

The discriminant of this quadratic in y is

$$D := -\mathrm{s}^4\mathrm{c}^2(4^{-1}\mathrm{c}^4(u + u_1)^2 + \mathrm{s}^2\mathrm{c}^2uu_1 - \mathrm{s}^4f^2)\,,$$

and from this we can conclude the following:

Lemma 1. (i) *If* $|u_1| \geq 2\mathrm{s}^2\mathrm{c}^{-2}f$ *then* $\Theta(u_1)$ *does not intersect* $\Theta(u)$ *for any* $u(\neq u_1)$ *of the same sign as* u_1*: one circle contains the other.*

(ii) *If* $|u_1| \leq \mathrm{s}^2\mathrm{c}^{-1}f$ *then* $\Theta(u_1)$ *intersects all circles* $\Theta(u)$ *for which* $|u - u_1| < \varepsilon$*, for some* ε *depending upon* u_1*.*

Thus in case (ii) we can expect osculation, and only in this case. Let $u \to u_1$ in (1), (2) to obtain the coordinates (x, y) of the two limiting positions of the points of intersection of the two circles. (In effect, the calculation has been to eliminate u from the equations $\theta(u) = 0$, $\theta'(u) = 0$.) Make a clockwise rotation through angle $\frac{1}{2}(\pi - \omega)$ from axes Oxy to new axes O$\lambda\mu$: there is an unexpected disappearance of the u_1 terms from the algebra, and we arrive at

$$\Gamma_1\colon \lambda^2 + \mu^2\mathrm{c}^{-2} = f^2\,.$$

This is the equation of an ellipse with eccentricity s, foci at F, H, and passing through G, I; it is the curve osculated by the circumcircles $\Theta(u)$ when $|u| \leq \mathrm{s}^2\mathrm{c}^{-1}f$. The points on the diagonal through I, G where $u = \pm\mathrm{s}^2\mathrm{c}^{-1}f$ lie between G and I. As the centre U(u) of $\Theta(u)$ moves from $u = +\mathrm{s}^2\mathrm{c}^{-1}f$ to $u = -\mathrm{s}^2\mathrm{c}^{-1}f$, the points of osculation (there are two of them) move from coincidence at I, around the ellipse in opposite directions to G. Since each circle osculates at two points, we deem this ellipse to be double.

All this calculation assumes that U is on the line $\lambda = 0$ containing IG; but of course there are similar outcomes for the line $\mu = 0$ along the shorter diagonal FH. Here $\mathrm{U}(u)$ has coordinates $x = -\mathrm{s}u$, $y = \mathrm{c}u$, and the osculating curve for the circles with centres $\mathrm{U}(u)$ on this line is found to be the ellipse

$$\Gamma_2 \colon \lambda^2 \mathrm{s}^{-2} + \mu^2 = f^2 \,,$$

having eccentricity c, foci at G, I, and passing through F, H. To summarize:

Theorem 1. *When Λ is a rhombus with sidelength f and angle ω, $0 < \omega \le \frac{1}{2}\pi$, the circumcircles $\Theta(u)$ of the cyclic quadrilaterals in $\mathcal{Q}(\Lambda)$ osculate an octic curve, consisting of the two ellipses Γ_1 and Γ_2, each counted twice. The osculation occurs for $|u| \le \mathrm{s}^2\mathrm{c}^{-1}f$, $|u| \le \mathrm{c}^2\mathrm{s}^{-1}f$ respectively. The ellipses have major semi-axes of equal lengths f and minor semi-axes of lengths $\mathrm{c}f$, $\mathrm{s}f$ respectively.*

Γ for a non-rhombus

Now suppose that Λ is a non-rhomboidal parallelogram. The problem is of an altogether greater level of difficulty. Let the sidelengths be f, g with $f > g$, the angle be ω, with $0 < \omega \le \frac{1}{2}\pi$. We take the same pairs of axes $\mathrm{O}x, \mathrm{O}y$ and $\mathrm{O}\lambda, \mathrm{O}\mu$ as before, so that $\lambda = \mathrm{s}x - \mathrm{c}y$, $\mu = \mathrm{c}x + \mathrm{s}y$, which reduces the equation of the locus of circumcentres $\mathrm{Cen}(\Lambda)$, a rectangular hyperbola, to

$$\lambda\mu = \mathrm{e}^2 \,, \quad \text{where } \mathrm{e} := \frac{\sqrt{(f^2 - g^2)\mathrm{sc}}}{2} \,. \tag{3}$$

See [1, Equation (3)]. We change to a parameter t (differently defined from u): the circumcircle $\Theta(t)$ with centre $\mathrm{T} = (\mathrm{e}t, \mathrm{e}t^{-1})$ on the hyperbola has equation $\psi(t) = 0$ where

$$\psi(t) := \lambda^2 + \mu^2 - 2e(\lambda t + \mu t^{-1}) - \mathrm{e}^2(\mathrm{s}^2\mathrm{c}^{-2}t^2 + \mathrm{c}^2\mathrm{s}^{-2}t^{-2}) - h^2 \,, \quad h^2 := \tfrac{1}{2}(f^2 + g^2) \,. \tag{4}$$

(See [1, Theorem 5].) To find the curve of osculation Γ we have to eliminate t from the two equations $\psi(t) = 0$, $\psi'(t) = 0$. This involves equating to zero their resultant, an 8×8 determinant (that is, the discriminant of ψ; see [6, pp. 83–88]). Such a determinant has 40 320 terms, but mercifully most of these are zero. The outcome, after heavy algebra (or more simply by use of Mathematica), is:

$$\begin{aligned} \mathrm{R} = {} & \mathrm{s}^2\mathrm{c}^2\zeta^8 + (\mathrm{c}^4\lambda^2 + \mathrm{s}^4\mu^2)\zeta^6 + \mathrm{s}^2\mathrm{c}^2(\lambda^2\mu^2 - 20\lambda\mu\mathrm{e}^2 - 8\mathrm{e}^4)\zeta^4 \\ & - 18\mathrm{e}^2[\mathrm{c}^4\lambda^3\mu + \mathrm{s}^4\lambda\mu^3 + 2\mathrm{e}^2(\mathrm{c}^4\lambda^2 + \mathrm{s}^4\mu^2)]\zeta^2 - 27\mathrm{e}^4\mathrm{c}^6\mathrm{s}^{-2}\lambda^4 \\ & - 27\mathrm{e}^4\mathrm{s}^6\mathrm{c}^{-2}\mu^4 + 16\mathrm{e}^8\mathrm{s}^2\mathrm{c}^2 - \mathrm{e}^2\mathrm{s}^2\mathrm{c}^2[16\lambda^3\mu^3 + 6\mathrm{e}^2\lambda^2\mu^2 + 48\mathrm{e}^4\lambda\mu] \\ = {} & 0 \,, \end{aligned}$$

where ζ^2 is written for $\lambda^2 + \mu^2 - \frac{1}{2}(f^2 + g^2)$, and an irrelevant constant factor has been omitted. Simplifying further by substituting for ζ^2 we eventually arrive at an octic equation in λ and μ, and the following result:

Theorem 2. *When Λ is not a rhombus, the curve of osculation of the family $\mathcal{C}(\Lambda)$ of circumcircles of the pantograph $\mathcal{Q}(\Lambda)$ referred to axes $\mathrm{O}\lambda\mu$ has equation $\mathrm{R} = 0$, where*

$$\mathrm{R} := \mathrm{R}_8 - \mathrm{R}_6 + \mathrm{R}_4 + \mathrm{R}_2 + \mathrm{R}_0 \,, \tag{5}$$

each R_j *denoting a homogeneous polynomial of degree* j *in* λ *and* μ*, as follows:*

$$\begin{aligned}
\mathrm{R}_8 &= \mathrm{c}^2\lambda^8 + (1+3\mathrm{c}^2-\mathrm{c}^4)\lambda^6\mu^2 + (3+2\mathrm{s}^2\mathrm{c}^2)\lambda^4\mu^4 + (1+3\mathrm{s}^2-\mathrm{s}^4)\lambda^2\mu^6 + \mathrm{s}^2\mu^8\,,\\
\mathrm{R}_6 &= h^2\mathrm{c}^2(\mathrm{s}^2+3)\lambda^6 + \mathrm{e}^2\mathrm{c}^2(20\mathrm{s}^2+18\mathrm{c}^2)\lambda^5\mu + h^2(3+8\mathrm{c}^2-5\mathrm{c}^4)\lambda^4\mu^2\\
&\quad + \mathrm{e}^2(20\mathrm{s}^2\mathrm{c}^2+18)\lambda^3\mu^3 + h^2(3+8\mathrm{s}^2-5\mathrm{s}^4)\lambda^2\mu^4\\
&\quad + \mathrm{e}^2\mathrm{s}^2(20\mathrm{c}^2+18\mathrm{s}^2)\lambda\mu^5 + h^2\mathrm{s}^2(\mathrm{c}^2+3)\mu^6\,,\\
\mathrm{R}_4 &= [3h^4(1-\mathrm{s}^4) + \mathrm{e}^4\mathrm{s}^{-2}\mathrm{c}^2(-27+18\mathrm{s}^2+\mathrm{s}^4)]\lambda^4 + h^2\mathrm{e}^2\mathrm{c}^2(40-22\mathrm{c}^2)\lambda^3\mu\\
&\quad + [3h^4\mathrm{s}^2\mathrm{c}^2(1+3\mathrm{s}^2\mathrm{c}^2) + \mathrm{e}^4(50\mathrm{s}^2\mathrm{c}^2-36)]\lambda^2\mu^2 + h^2\mathrm{e}^2\mathrm{s}^2(40-22\mathrm{s}^2)\lambda\mu^3\\
&\quad + [3h^4(1-\mathrm{c}^4) + \mathrm{e}^4\mathrm{s}^2\mathrm{c}^{-2}(-27+18\mathrm{c}^2+\mathrm{c}^4)]\mu^4\,,\\
\mathrm{R}_2 &= [-h^6\mathrm{c}^2(1+3\mathrm{s}^2) + 4h^2\mathrm{e}^4\mathrm{s}^2\mathrm{c}^4(9-5\mathrm{s}^2)]\lambda^2 - 4\mathrm{s}^2\mathrm{c}^2(5\mathrm{e}^2h^4+12e^6)\lambda\mu\\
&\quad + [-h^6\mathrm{s}^2(1+3\mathrm{c}^2) + 4h^2\mathrm{e}^4\mathrm{s}^4\mathrm{c}^2(9-5\mathrm{c}^2)]\mu^2\,,\\
\mathrm{R}_0 &= \mathrm{s}^2\mathrm{c}^2(h^4-4\mathrm{e}^4)^2.
\end{aligned}$$

Here e, h *are as defined in* (3), (4) *and* $\mathrm{s} = \sin\frac{1}{2}\omega$, $\mathrm{c} = \cos\frac{1}{2}\omega$.

The expression R is homogeneous of degree 8 in λ, μ, h and e, with 21 terms in various powers of λ and μ. Evidently R possesses certain symmetries. For example, $\mathrm{R}(\lambda,\mu) = \mathrm{R}(-\lambda,-\mu)$, and (if we show ω as a variable in R) $\mathrm{R}(\lambda,\mu,\omega) = \mathrm{R}(\mu,\lambda,\pi/2-\omega)$.

To find the point where, for a given t, the circle $\Theta(t)$ touches the osculating curve, we have to solve the equations $\psi(t) = 0$, $\psi'(t) = 0$ for λ,μ. This involves solving a simultaneous pair of equations of the form $\mathrm{X}^2+\mathrm{Y}^2 = \mathrm{R}^2$, $\mathrm{AX}+\mathrm{BY} = \mathrm{K}$, where X,Y are linear forms in λ and μ. Doing this, we find that the solution is real if and only if the expression

$$\Delta(t) := -t^4\mathrm{e}^2\mathrm{s}^2\mathrm{c}^{-4} + t^2h^2 + 3\mathrm{e}^2\mathrm{s}^{-2}\mathrm{c}^{-2} + t^{-2}h^2 - t^{-4}\mathrm{e}^2\mathrm{c}^2\mathrm{s}^{-4}$$

is positive or zero. This expression has the same sign as $\mathrm{s}^4\mathrm{c}^4t^4\Delta(t)$; so write $v = t^2$ and consider the quartic polynomial

$$\mathrm{D}(v) := -v^4\mathrm{e}^2\mathrm{s}^6 + v^3\mathrm{s}^4\mathrm{c}^4h^2 + 3v^2\mathrm{e}^2\mathrm{s}^2\mathrm{c}^2 + v\mathrm{s}^4\mathrm{c}^4h^2 - \mathrm{e}^2\mathrm{c}^6\,.$$

By examining the properties of D and its derivatives we find that the graph of $w = \mathrm{D}(v)$ cuts the w-axis at $w = -\mathrm{e}^2\mathrm{c}^6$, that $w \to -\infty$ as $v \to \infty$, and there is a single maximum of D on $v > 0$. Moreover $\mathrm{D}(v) > 0$ when $v = (\mathrm{cs}^{-1})^{3/2}$. Thus D has exactly two positive zeros, at say $v = \alpha^2, \beta^2$, between which it is positive, and therefore the simultaneous equations $\psi(t) = 0$, $\psi'(t) = 0$ have a real solution for λ, μ when $\alpha < |t| < \beta$. We conclude:

Lemma 2. *The circles* $\Theta(t)$ *osculate the octic curve* (5) *when* $-\beta \le t \le -\alpha$ *and when* $\alpha \le t \le \beta$*, but for no other values of* t*. Here* α, β *are two numbers satisfying*

$$0 < \alpha < (\mathrm{cs}^{-1})^{3/4} < \beta\,.$$

The formulae for α and β are complicated.

The nature of the octic curve Γ

We can discover the shape of Γ by the use of *Mathematica*. The following program gives a plot of Γ and its originating parallelogram Λ, for arbitrary values of f, g and ω, where as usual we assume $0 < g < f$ and $0 < \omega < \pi/2$.

```
p1 = e^2 c^4 + 2 e s^2 c^2 μ t - s^2 c^2 (λ^2 + μ^2 - h^2) t^2
     + 2 e s^2 c^2 λ t^3  + e^2 s^4 t^4;
q1 = - e^2 c^4 - e s^2 c^2 μ t + e s^2 c^2 λ t^3 + e^2 s^4 t^4;
p2 = Expand[p1/.{λ -> s x - c y, μ -> c x + s y,
     e -> Sqrt[(f^2 - g^2) s c]/2, h -> Sqrt[(f^2 + g^2)/2]}];
q2 = Expand[q1/.{λ -> s x - c y, m -> c x + s y,
     e -> Sqrt[(f^2 - g^2) s c]/2, h -> Sqrt[(f^2 + g^2)/2]}];
r1 = Resultant[p2, q2, t];
r2 = r1/.{s -> Sin[ω/2], c -> Cos[ω/2]};
r3 = FullSimplify[r2];
r4 = r3*(-2199023255552/((f^2 - g^2)^4 Sin[ω]^18));
Manipulate[Show[ContourPlot[r4, {x, -30, 30}, {y,- 30,30}, Axes -> True,
   AxesLabel -> {x,y}, Frame -> False, Contours -> {0},
   ContourStyle -> {Thick, Black}], ListPlot[{{(-f + g Cos[ω])/2,
   g Sin[ω]/2},{(f + g Cos[ω])/2, g Sin[ω]/2},{(f - g Cos[ω])/2,
   - g Sin[ω]/2},{(- f - g Cos[ω])/2,- g Sin[ω]/2},{(-f + g Cos[ω])/2,
   g Sin[ω]/2}}, PlotStyle -> {Thickness[0.005], Black},
   PlotJoined -> True]], {ω, 0., Pi/2}, {f,0,20}, {g,0,f}]
```

Here p1, q1 are the functions $\psi(t)$, $\psi'(t)$ respectively, shorn of irrelevant factors; p2, q2 express these functions in terms of x and y, and substitute for e and h. Then r1 is the resultant, eliminating parameter t; this has 1326 summands. Expression r2 substitutes the trigonometric values of s and c; r3 finds a constant factor, allowing this to be divided out for r4.

The equation $\mathrm{R} = 0$ defining Γ referred to axes Oxy is then essentially the equation $\mathrm{r4} = 0$. To plot the implicit equation $\mathrm{R(x, y)} = 0$, we ask for the contour $z = 0$ of the surface $z = \mathrm{R}(x, y)$. It may be necessary to substitute for some labels such as r1 their full expressions. ListPlot[...] gives the parallelogram Λ. The two graphs can be manipulated together using sliders for ω, f and g. Fixing f, we get a two-parameter family of octics, depending on parameters f/g and ω from the dimensions for the parallelogram.

Figure 2 shows Γ for three choices of the shape of Λ. When g is sufficiently less than f (first plot), Γ takes the form of two identical but rotated smooth egg-shaped circuits, overlapping, so that a transversal line cuts the curve in at most 4 real points, a general pencil of real lines through an external point contains 4 tangents, there are 2 crunodes, no cusps, 2 double tangents, and no inflexions. As g nears f the curve becomes more erratic (middle plot), until the curve seemingly approximates to a four-bean quartic[2] having 4 circuits and 28 real bitangents, and thus approximates to the pair of ellipses which we met aready for the case when $g = f$ and Λ is a rhombus. Γ always passes through the vertices of Λ.

[2]Presumably the quartic is doubled. For four-bean quartics see [2, Chapter XIX, p. 342].

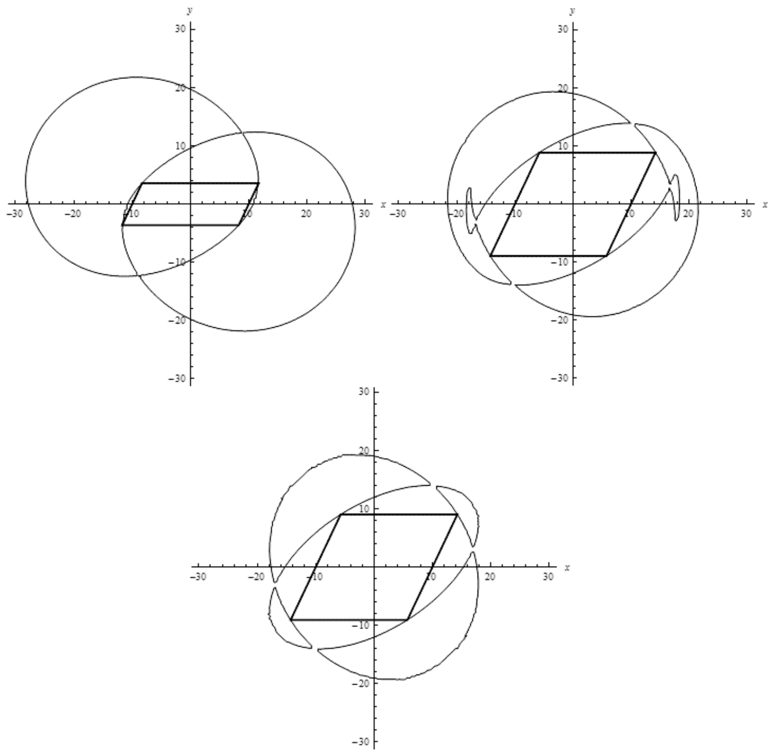

Figure 2. Three depictions of Γ, for $f = 20$ and $\omega = 1.28$ radians. They are the cases $g = 8$, $g = 19.68$ and g slightly less than 20, respectively.

The process of osculation can be observed by a second Mathematica program. See Figure 3. The interval $[\alpha, \beta]$ of Lemma 2 in the stated case is approximately $[0.5, 6]$. As t increases from 0.5, the osculation point starts somewhere about $x = -6$, $y = -20$, it splits immediately into two points which move in opposite directions around the righthand circuit of Γ until they meet again when $\mathrm{t} \approx 6$; thereafter $\Theta(\mathrm{t})$ and Γ lose contact.

The values for the Plücker numbers of Γ implied by the previous paragraph do not satisfy Plücker's equations[3]. We must conclude either that the equation of Γ contains a squared factor, so that part or all of Γ is described twice, or that there are more points on the curve than are shown in the diagrams. A study of the Mathematica plots fails to reveal any extra real parts to the curve. Separately, these plots suggest that R factors in the form $\mathrm{R}(x, y) = \mathrm{M}(x, y).\mathrm{M}(-x, -y)$, where M would of course be a quartic; but Mathematica finds no factors other than constants, even over $\mathbb{C}[x, y]$. So there remain some unresolved questions about Γ.

The rectangular hyperbola Cen(Λ) is the *evolute* of Γ, being the locus of its centre of curvature. It is not asserted that all points of Γ are points of osculation; though

[3]For a discussion of Plücker numbers and equations see [2, Chapter VIII].

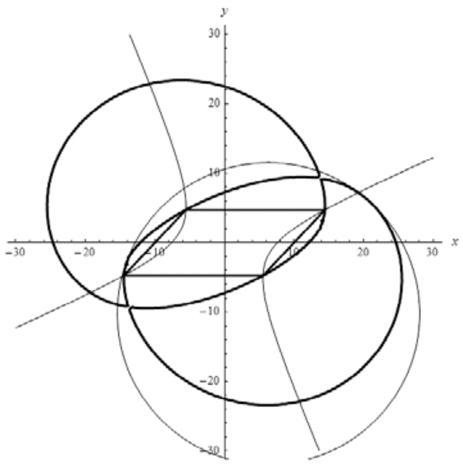

Figure 3. Osculation, the particular case when $f = 20$, $g = 13$, $\omega = 0.815$, showing the circle $\Theta(t)$ for $t = 2.6$.

this is evidently the case when g is much less than f, from the Mathematica plots. The circles are bitangent circles to Γ; the minimum of their radii is f.

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References

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