Puzzle corner

Norman Do*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of \$50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 7 is 1 July 2008. The solutions to Puzzle Corner 7 will appear in Puzzle Corner 9 in the September 2008 issue of the *Gazette*.

Physicists and chemists

A group of N mathematicians and N physicists sit around a circular table. Some of them always tell the truth, while the others always lie. It is known that the number of liars among the mathematicians and the number of liars among the physicists are equal. Everyone is asked, 'What is your right-hand neighbour?' and they all reply, 'Physicist.' Prove that N must be even.

Sums of alternating sums

If A is a finite set of positive integers, we form its alternating sum by arranging its elements in decreasing order of magnitude and alternately adding and subtracting them. For example, the alternating sum of $X = \{11, 6, 17, 1, 9, 18, 13\}$ is $18 - 17 + 13 - 11 + 9 - 6 + 1 = 7$. What is the sum of the alternating sums over all subsets of $\{1, 2, \ldots, n\}$?

Rational or irrational?

For a positive integer k, let A_k be the real number between 0 and 1 formed by writing the perfect kth powers in order after the decimal point. For example, $A_1 =$

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0.123456789101112... and $A_2 = 0.149162536496481100121144\ldots$. Are there any values of k for which A_k is rational?

Polygons and rectangles

Prove that every convex polygon of area 1 is contained in a rectangle of area 2.

Bonus: Is there a positive real number $R < 2$ such that every convex polygon of area 1 is contained in a rectangle of area R? If so, then what is the smallest possible value of R?

The broken calculator

Photo: Tibor Fazakas

A calculator is broken so that the only buttons that still work are sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. The calculator performs all calculations in radians and with infinite precision. If the display initially shows the number 0, prove that it is possible to produce any positive rational number by pressing some finite sequence of buttons.

Chessboard puzzles

As many of you will know, a chessboard consists of an 8×8 grid of squares, coloured alternately black and white. Chessboards are involved in a vast number of mathematical puzzles, three of which are presented here.

(1) Let $B_1, B_2, \ldots, B_{32}$ be the centres of the black squares of a chessboard and let $W_1, W_2, \ldots, W_{32}$ be the centres of the white squares. Prove that for every point P on the chessboard,

$$\overline{PB}_1^2 + \overline{PB}_2^2 + \cdots + \overline{PB}_{32}^2 = \overline{PW}_1^2 + \overline{PW}_2^2 + \cdots + \overline{PW}_{32}^2.$$

(2) If the squares of a chessboard are labelled with the numbers from 1 to 64 with one number in each square, then we define the *score* of the labelling to be the largest difference between two labels in squares which share a common side or a common vertex. What is the smallest possible score that can be obtained?

(3) I would like to tile the squares of a chessboard with twenty-one 3×1 rectangles and one 1×1 square. In which squares of the chessboard can the 1×1 square be placed?

Solutions to Puzzle Corner 5

The \$50 book voucher for the best submission to Puzzle Corner 5 is awarded to Konrad Pilch.

Pricey pills

Justin Ghan: The sick patient should take out another Xenitec pill from the bottle, so that they have two of each on the table. They can then divide each pill into halves, take half of each of the four pills today, and the remaining four halves tomorrow.

Marching band


Photo: Holger Selover-Stephan

Mitch Wheat: Denote the height, in centimetres, of the band member who ends up in row i and column j after the two sorts by $b_{i,j}$. Suppose that the column sort puts the columns in nondecreasing order from left to right and that the row sort puts the rows in nondecreasing order from top to bottom. If the columns are not still in nondecreasing order after the row sort, then there exist indices i, j, k such that $b_{i,k} > b_{j,k}$ for $i < j$. Since rows i and j are sorted properly, we have the following chain of inequalities, where n denotes the number of columns.

$$b_{j,1} \le b_{j,2} \le \ldots \le b_{j,k} < b_{i,k} \le b_{i,k+1} \le \ldots \le b_{i,n} .$$

Therefore, we have $n+1$ people sorted in order. By the pigeonhole principle, two of these people must have been in the same column prior to the rows being sorted. Clearly, one of these people is from row i while the other is in row j and we call these people A and B, respectively. By the previous chain of inequalities, we know that A is taller than B, contradicting the assumption that the column containing A and B was in nondecreasing order. Therefore, we conclude that the columns are still in nondecreasing order of height after the two sorts.

Silly soldiers

Konrad Pilch: The maximum time necessary for the soldiers to stabilise their positions is $n-1$ seconds. Rather than soldiers turning around whenever they are facing each other, assume that they step forward and take each other's position. Hence, each soldier only moves forward and never changes orientation. Define a left soldier to be one who faces left and a right soldier to be one who faces right. Note that a stable position must consist of a number of left soldiers on the left followed by a number of right soldiers on the right. Note also that the left soldiers preserve their order and so do the right soldiers.

We now prove by induction that it takes at most $n-1$ seconds for stability to occur. The base cases $n = 1$ and $n = 2$ are clear. Now suppose that it is true for up to k soldiers and consider $k+1$ soldiers in a line. If the leftmost soldier is already facing left, then they never move and so the configuration will stabilise in at most $k-1$ seconds. The situation is similar if the rightmost soldier is already facing right. Therefore, assume that the leftmost soldier, say A, is facing right while the rightmost soldier, say D, is facing left. Also, let the second leftmost right soldier be B and the second rightmost left soldier be C.

Now if we ignore A, then the remaining k soldiers will stabilise in at most $k-1$ seconds and D will be to the left of B. Similarly, if we ignore D, then the remaining k soldiers will stabilise in at most $k-1$ seconds and A will be to the right of

C. Therefore, after $k-1$ seconds, it must be the case that D is to the left of B while A is to the right of C. Also, A is to the left of B and D is to the right of C. In fact, if we represent left soldiers by $\Leftarrow$ and right soldiers by $\Rightarrow$, then the situation after $k-1$ seconds must be one of the following two cases.

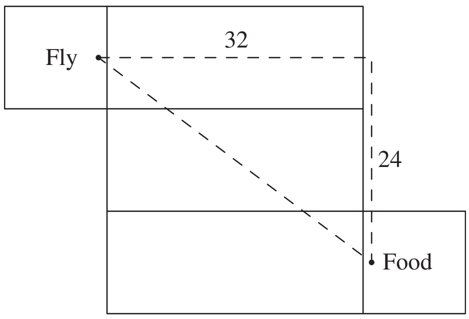
$$\Leftarrow\Leftarrow\cdots\Leftarrow\overset{\Leftarrow}{C}\;\overset{\Leftarrow}{D}\;\overset{\Rightarrow}{A}\;\overset{\Rightarrow}{B}\Rightarrow\cdots\Rightarrow\Rightarrow$$
$$\Leftarrow\Leftarrow\cdots\Leftarrow\overset{\Leftarrow}{C}\;\overset{\Rightarrow}{A}\;\overset{\Leftarrow}{D}\;\overset{\Rightarrow}{B}\Rightarrow\cdots\Rightarrow\Rightarrow$$

The first case is already stable while the second requires one more second to stabilise. So, in both cases, stability occurs in at most k seconds. Therefore, by induction, any sequence of n soldiers will stabilise in at most $n-1$ seconds. Note that if B or C does not exist, then the proof is essentially identical. Also, we provide the following example of n soldiers which takes $n-1$ seconds to stabilise.

$$\Rightarrow\underbrace{\Leftarrow\Leftarrow\cdots\Leftarrow\Leftarrow}_{n-1}$$

The lazy fly

John Harper: The following diagram is part of a net from which the room can be folded. The distance from the fly to the food in this picture can be calculated using Pythagoras' theorem to be $\sqrt{24^2+32^2}=40$ feet. This gives a path of length 40 feet which can be shown to be minimal by considering all possible nets for the room.



Dinner party handshakes

Sam Krass: Each person must shake hands with at least 0 people and at most 18 people. Therefore, the 19 different answers must have been the integers from 0 to 18. So there is someone who shook hands with 18 people, and their spouse must be the one who shook hands with 0 people. We can now remove this pair and reduce each person's handshake tally by one. This leaves us with an analogous situation with 9 couples. Working inductively, we can once more remove the couple consisting of a person who shook hands with everyone, apart from their spouse, and their spouse who shook hands with no one. Hence, we can remove nine couples, leaving only the inquisitor and his wife at the dinner party. In this final situation, his wife must have shaken zero hands. Retracing our steps backwards, we deduce that his wife must have shaken hands precisely 9 times at the dinner party.

Counting the digits

Peter Pleasants: The number of digits of a number N in base b is $\lfloor \log_b N \rfloor + 1$. Therefore, the numbers of digits, b_k and c_k, of the kth numbers in lists B and C are $\lfloor k \log_2 10 \rfloor + 1$ and $\lfloor k \log_5 10 \rfloor + 1$. If we write $x = \log_2 10$ and $y = \log_5 10$, then x and y are irrational numbers with

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{\log_2 10} + \frac{1}{\log_5 10} = \frac{\log 2}{\log 10} + \frac{\log 5}{\log 10} = 1.$$

We now prove a result known as Beatty's theorem from which the problem follows. It states that, if positive irrational numbers x and y satisfy $(1/x) + (1/y) = 1$, then the two sequences $\lfloor x \rfloor, \lfloor 2x \rfloor, \lfloor 3x \rfloor, \ldots$ and $\lfloor y \rfloor, \lfloor 2y \rfloor, \lfloor 3y \rfloor, \ldots$ together contain every positive integer exactly once. Consider the following inequalities for some positive integer n.

$$kx < n \Rightarrow (n-k)y > n$$
$$(k+1)x > n+1 \Rightarrow (n-k)y < n+1$$

Together, they imply that if the sequence $x, 2x, 3x, \ldots$ steps over the interval $(n, n+1)$, then the sequence $y, 2y, 3y, \ldots$ steps into it. Also, note that no term in either sequence can ever be an integer, due to the irrationality of x and y. Furthermore, since x and y are both greater than 1, neither can step into the same interval twice. Finally, since one of x and y must be less than 2, the corresponding sequence starts in the interval $(1, 2)$. Putting all these facts together, we deduce Beatty's theorem.

Tennis anyone?

For the first time in Puzzle Corner history, there have been no correct solutions submitted to a problem. The following are provided by your humble author.

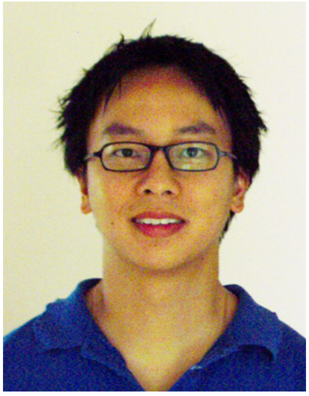
(1) In order to determine the tournament's winner, the remaining 999 players must lose one match each. Since there is exactly one loser in every match, the number of matches must be 999.

(2) Note that the men's Wimbledon final is played to the best of five sets while the women's Wimbledon final is played to the best of three sets. Therefore, if you are a man, then you would want to be leading 2–0 in sets, drawn 6–6 in games and leading 6–0 in the tie break. Then you will win the tournament if you can secure any of the following six points. On the other hand, if you are a woman, then you would want to be leading 1–0 in sets, drawn 6–6 in games and leading 6–0 in the tie break.

(3) Amazingly, the probability of Alex winning is the same, whether he chooses the alternating serves (AS) scheme or the winner serves (WS) scheme! Suppose that Alex and Bobbi continue to play for 23 games in the AS scheme. Note that Alex will have served 12 times and Bobbi will have served 11 times. The winner is simply the one who wins the majority of these 23 games, independent of the results of the games played after one player has won the match.

    Now suppose that Alex and Bobbi continue to play for 23 games in the WS scheme, where the loser serves out the remaining games after one player has reached 12 games. Again, note that Alex will have served 12 times and Bobbi

will have served 11 times. And again, the winner is simply the one who wins the majority of these 23 games, independent of the results of the games played after one player has won the match.
Since the probability of a win depends only on the server, the probability that Alex beats Bobbi is the same in either serving scheme. In fact, it is simply the probability of Alex winning the majority of games in a 23 game match where Alex serves 12 times and Bobbi serves 11 times.



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