MVT rules OK!

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One of the favourite problems of first-year calculus is testing whether a function defined by different formulae on different parts of the domain is differentiable at the 'break points'. A typical function may be given by $f(x) = 2\sin x + 1$ to the left of 0, and by $f(x) = 3x^2 + 2x + 1$ at 0 and to the right of it. The expected solution:

$$x < 0\colon \quad \frac{f(x) - f(0)}{x - 0} = \frac{2\sin x}{x} \to 2 \quad \text{as } x \to 0-,$$
$$x > 0\colon \quad \frac{f(x) - f(0)}{x - 0} = \frac{3x^2 + 2x}{x} = 3x + 2 \to 2 \quad \text{as } x \to 0+\,.$$

The two one-sided derivatives exist and are equal, hence f has a two-sided derivative at 0, equal to 2.

However, this is not how most students go about it. They take $f'(x) = 2\cos x$ on the left with the limit 2 as $x \to 0-$, and $f'(x) = 6x + 2$ on the right with the limit 2 as $x \to 0+$. Then they conclude that

$$f'(0) = \lim_{x\to 0} f'(x) = 2.$$

But this may fail when f has a discontinuity at the break point. Set $g(x) = 2\sin x + 5$ for $x < 0$ and $g(x) = 3x^2 + 2x - 1$ for $x \geq 0$. The preceding argument seemingly goes through, yet the function g is not even continuous at 0. But what happens when in addition f is continuous at the break point? Are the students right in using this method, or can we come up with a counterexample?

Trying to produce a counterexample we may use MAPLE. Say we want to find a function whose derivative is $g(x) = 2x\sin(1/x)$ for $x \neq 0$. According to MAPLE, one such function is

$$f(x) = \begin{cases} x^2\sin(1/x) + x\cos(1/x) + \mathrm{Si}(1/x), & x > 0, \\ x^2\sin(1/x) + x\cos(1/x) + \mathrm{Si}(1/x) + \pi, & x < 0, \end{cases} \tag{1}$$

where $\mathrm{Si}(u) = \int_0^u t^{-1}\sin t\,dt$. The first two terms on the right-hand side of (1) go to 0 as $x \to 0$, and MAPLE tells us that the limit of $\mathrm{Si}(t)$ is $\pm\frac{1}{2}\pi$ as $t \to \pm\infty$. Hence $\lim_{x\to 0} f(x) = \frac{1}{2}\pi$, and we set $f(0) = \frac{1}{2}\pi$ to achieve the continuity of f at 0. At any point $x \neq 0$ we have $f'(x) = 2x\sin(1/x)$, so that $\lim_{x\to 0} f'(x) = 0$ by the sandwich rule. But does the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \tag{2}$$

exist? MAPLE says the limit is *undefined*! Do we have a counterexample?

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Let us look more closely at the method used by the students and try to formulate it precisely:

Conjecture. Suppose that a real valued function f is continuous at a point $a \in \mathbb{R}$ and that

$$\lim_{x \to a} f'(x) = c \in \mathbb{R}.$$

Then f is differentiable at a, and $f'(a) = c$.

True or false? Calculus books are consulted in vain, but various functions seem to confirm the hunch that — uncharacteristically — the students may be right. Many attempts at a proof lead nowhere. The deltas seem to tamely follow the epsilons, only to run away at the last moment. Then a moment of unsurpassed clarity, the classical *eureka* moment that will be remembered for ever: When everything else fails, *think the mean value theorem*!

Proof of the conjecture. The existence of the limit means that $f'(x)$ exists for all x satisfying $0 < |x - a| < \eta$ for some $\eta > 0$. If $a < x < a + \eta$, then f is continuous in $[a, x]$ and differentiable in (a, x). By the mean value theorem there exists a point ξ (depending on x) such that $a < \xi < x$ and

$$\frac{f(x) - f(a)}{x - a} = f'(\xi).$$

Clearly $\xi \to a+$ as $x \to a+$, and

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a+} f'(\xi) = \lim_{\xi \to a+} f'(\xi) = c,$$

that is, the derivative of f at a from the right is c. A symmetrical argument shows that the derivative of f at a from the left is also c. *Done*!

As the first aplication of our magnificent theorem (conjecture no more!) we note that for the function f defined by (1) the limit in (2) exists and equals 0. Eat your heart out, MAPLE!

⋆ ⋆ ⋆

Postscript. For an account of the mean value theorem (MVT to the initiated) with an elegant novel proof see [1]. While not widely known, our 'Conjecture' is tucked away as exercise in some good books on analysis, for instance as Exercise 4 on page 183 of Stromberg's text [2].

References

[1] Koliha, J.J. and Zhang, P. (2007). Rolle to Cauchy. *Gaz. Aust. Math. Soc.* **34,** 210–211.

[2] Stromberg, K. (1982). *Introduction to Classical Real Analysis.* Wadsworth International Group, Belmont, CA.