The case of the mysterious sevens

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Abstract

We give a simple q–series proof of a partition theorem of Farkas and Kra.

In 2003, Hershel Farkas toured Australia, and gave a talk in several places, in which he presented a partition theorem discovered by him and Irwin Kra [1], and proved by them using theta-functions. Their theorem is as follows.

Theorem 1. *If S is the set consisting of all the positive integers in black together with multiples of 7 in red then the number of partitions of $2N+1$ into distinct odd elements of S is equal to the number of partitions of $2N$ into distinct even elements of S.*

Farkas issued a challenge to his audience, to find a simpler proof (a generating function proof or a combinatorial proof) of their theorem. I rose to his challenge [3], as did Warnaar [4], who generalised their result. I have since found a simpler version of my proof, and will present it here.

It is easy to see that the theorem is equivalent to the q-series result

$$\mathbf{O}\left(\prod_{n=1}^{\infty}(1+q^{2n-1})(1+q^{14n-7})\right) = q\prod_{n=1}^{\infty}(1+q^{2n})(1+q^{14n})\,, \tag{1}$$

where $\mathbf{O}$ denotes the *odd part* of the series.

If we multiply by $\prod_{n=1}^{\infty}(1-q^{4n})(1-q^{28n})$ and use the special cases of Jacobi's triple-product identity [2, (19.9.1)] with $(k,l) = (2,1)$, $(14,7)$, $(4,2)$, $(28,14)$,

$$\prod_{n=1}^{\infty}(1+q^{2n-1})(1-q^{4n}) = \prod_{n=1}^{\infty}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n}) = \sum_{n=-\infty}^{\infty} q^{2n^2+n}\,,$$

$$\prod_{n=1}^{\infty}(1+q^{14n-7})(1-q^{28n}) = \sum_{n=-\infty}^{\infty} q^{14n^2+7n}\,,$$

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^{2n})(1-q^{4n}) &= \prod_{n=1}^{\infty}(1+q^{4n-2})(1+q^{4n})(1-q^{4n}) \\ &= \prod_{n=1}^{\infty}(1+q^{8n-6})(1+q^{8n-2})(1-q^{8n})\end{aligned}$$

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$$= \sum_{n=-\infty}^{\infty} q^{4n^2+2n},$$

and

$$\prod_{n=1}^{\infty}(1+q^{14n})(1-q^{28n}) = \sum_{n=-\infty}^{\infty} q^{28n^2+14n},$$

(note that all series and product statements and manipulations are valid in $|q| < 1$), we find that (1) is equivalent to

$$\mathbf{O}\left(\sum_{m,n=-\infty}^{\infty} q^{2m^2+m+14n^2+7n}\right) = q\sum_{m,n=-\infty}^{\infty} q^{4m^2+2m+28n^2+14n}. \tag{2}$$

But (2) is easy to prove!

We have

$$\sum_{m,n=-\infty}^{\infty} q^{2m^2+m+14n^2+7n} = \sum_{m,n=-\infty}^{\infty} q^{((m+7n+2)^2+7(m-n)^2-4)/4}$$
$$= \sum_{u-v\equiv 2 \pmod 8} q^{(u^2+7v^2-4)/4}.$$

If u and v are even and differ by $2 \pmod 8$, $(u^2+7v^2-4)/4$ is even, so these terms do not contribute to the odd part. Otherwise, we can set $(u,v) = (8k+1, -8l-1)$, $(8k+3, 8l+1)$, $(-8k-1, -8l-3)$ and $(-8k-3, 8l+3)$ in turn, and find

$$\mathbf{O}\left(\sum_{m,n=-\infty}^{\infty} q^{2m^2+m+14n^2+7n}\right)$$
$$= \sum_{k,l=-\infty}^{\infty} q^{16k^2+4k+112l^2+28l+1} + \sum_{k,l=-\infty}^{\infty} q^{16k^2+4k+112l^2+84l+15}$$
$$+ \sum_{k,l=-\infty}^{\infty} q^{16k^2+12k+112l^2+28l+3} + \sum_{k,l=-\infty}^{\infty} q^{16k^2+12k+112l^2+84l+17}$$
$$= q\left(\sum_{-\infty}^{\infty} q^{16k^2+4k} + q^2\sum_{-\infty}^{\infty} q^{16k^2+12k}\right)\left(\sum_{-\infty}^{\infty} q^{112l^2+28l} + q^{14}\sum_{-\infty}^{\infty} q^{112l^2+84l}\right)$$
$$= q\sum_{m,n=-\infty}^{\infty} q^{4m^2+2m+28n^2+14n},$$

as required.

References

[1] Farkas, H.M. and Kra, I. (2000). Partitions and theta constant identities. *The mathematics of Leon Ehrenpreis, Contemporary Mathematics* **251,** 197–203.

[2] Hardy, G.H. and Wright, E.M. (1960). *An Introduction to the Theory of Numbers*, 4th edn. Clarendon, Oxford.

[3] Hirschhorn, M.D. (2006). The case of the mysterious sevens. *International J. Number Theory* **2,** 213–216.

[4] Warnaar, S.O. (2005). A generalization of the Farkas–Kra partition theorem for modulus 7. *J. Comb. Theory Ser. A* **110,** 43–52.