Counting paths in a grid

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Abstract

We consider a problem of counting the total number of paths from a cell in row 1 to a cell in row m of an $m \times n$ grid of cells, with restrictions on the permissible moves from cell to cell. The problem arises in the context of counting the total number of train paths through a rail network. We develop a recurrence relation, which we solve by means of a generating function. The result is interpreted combinatorially.

Problem statement

We consider the problem of counting the total number of paths from a cell in row 1 to a cell in row m of an $m \times n$ grid of cells. The permissible moves from cell (i, j) are to $(i, j + 1)$, $(i + 1, j)$ or $(i + 1, j + 1)$. Such a path will therefore begin in cell $(1, p)$ and end in (m, q), for some p and q such that $1 \le p \le q \le n$.

A possible path is illustrated in Figure 1, where each cell on the path is represented by a 1, and each other cell by a 0. In this representation the three move types are, respectively, horizontal to the right, vertically down, and diagonally to the right and down.

	1	·	·	p	·	·	·	·	·	q	·	n
1	0	0	0	1	0	0	0	0	0	0	0	0
·	0	0	0	1	1	0	0	0	0	0	0	0
·	0	0	0	0	0	1	0	0	0	0	0	0
·	0	0	0	0	0	0	1	1	1	1	0	0
·	0	0	0	0	0	0	0	0	0	1	0	0
·	0	0	0	0	0	0	0	0	0	1	0	0
·	0	0	0	0	0	0	0	0	0	1	0	0
m	0	0	0	0	0	0	0	0	0	1	0	0

Figure 1. One possible path in the $m \times n$ grid.

We define $\mathcal{S}_{m,n}$ to be the set of such paths and let $P_{m,n}$ be the count of them. We find and solve a recurrence relation for $P_{m,n}$ and interpret the solution combinatorially.

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Application

This problem arises when considering the total number of possible train paths through a rail network, where the paths are defined not only by the track segments included but also by the timing of movements from segment to segment. The simple version discussed here is the case where all segments are included and need to be traversed in some particular order, and the only distinction between paths is the timing of the movements.

Each track segment in the rail network is represented in the grid by a row and each time period in the scheduling horizon (typically discretised in five-minute intervals) is represented by a column. A 1 in cell (i, j) indicates that track segment i is occupied by the train during time period j.

The three permissible move types represent, respectively, the train staying on the same segment in the next time period, moving to the next segment within the same time period, and moving to the next segment on commencement of the next time period.

We have some interest in the order of magnitude for $P_{m,n}$ for large values of m and n. We have formulated the train timetabling problem as a set covering model. The variables of the integer linear program represent different feasible train paths through the rail network. In the optimal solution, one path is selected for each train so that the total cost of the paths chosen is minimised.

If all possible paths for each train are included in the set covering model, then the solution is guaranteed to be optimal. Intuitively, we may already conclude that there are too many different possibilities for this approach to be viable. This is confirmed by the solution to the recurrence relation established later. Instead, to solve our integer program, we use a column generation approach in which new columns (train paths) are only generated as required.

Recurrence relation and solution

In this section we demonstrate that the recurrence relation for the number of paths, $P_{m,n}$, is

$$P_{m,n} = 2P_{m,n-1} - P_{m,n-2} + P_{m-1,n} - P_{m-1,n-2} \qquad m \geq 2, n \geq 3 \tag{1}$$

with boundary values $P_{1,n} = n(n+1)/2$ and $P_{m,1} = 1,\ P_{m,2} = 2m + 1,\ m \geq 2$.

Consider the subset $\mathcal{R}_{m,n}$ of $\mathcal{S}_{m,n}$ consisting of all the paths that begin in cell $(1, 1)$. The number of paths in $\mathcal{R}_{m,n}$ is clearly $P_{m,n} - P_{m,n-1}$, since those paths that are in $\mathcal{S}_{m,n} - \mathcal{R}_{m,n}$ must begin in one of the other $n - 1$ cells in row 1, and so are paths in an $m \times (n - 1)$ grid. Define the cardinality of $\mathcal{R}_{m,n}$ to be $a_{m,n}$. Thus

$$a_{m,n} = P_{m,n} - P_{m,n-1} \quad \text{for } m \geq 1, n \geq 2\,. \tag{2}$$

On the other hand, $a_{m,p}$ for $1 \leq p \leq n$ is also the number of paths in an $m \times n$ grid that commence in cell $(1, n - p + 1)$, so

$$P_{m,n} = \sum_{p=1}^{n} a_{m,p}\,. \tag{3}$$

The number of paths that commence in cell $(1, p)$, therefore, is $a_{m,n-p+1}$ for $1 \le p \le n$ so $P_{m,n}$ may also be expressed as

$$P_{m,n} = \sum_{p=1}^{n} a_{m,n-p+1} \,. \tag{4}$$

Now the paths in $\mathcal{R}_{m,n}$ can be partitioned into three subsets, according to the direction of the first move from $(1, 1)$.

If the first move is to $(1, 2)$, then the remainder of the path can be thought of as one of the $a_{m,n-1}$ paths that begin in cell $(1, 1)$ of an $m \times (n-1)$ grid (members of $\mathcal{R}_{m,n-1}$).

If the first move is to $(2, 1)$, then the remainder of the path can be thought of as one of the $a_{m-1,n}$ paths in $\mathcal{R}_{m-1,n}$.

If the first move is to $(2, 2)$, then the remainder of the path can be thought of as one of the $a_{m-1,n-1}$ paths in $\mathcal{R}_{m-1,n-1}$.

This covers all of the possibilities, so

$$a_{m,n} = a_{m,n-1} + a_{m-1,n} + a_{m-1,n-1} \quad \text{for } m \ge 2, n \ge 2. \tag{5}$$

Thus, from equation (2), the recurrence relation for P is

$$\begin{aligned} P_{m,n} - P_{m,n-1} = P_{m,n-1} &- P_{m,n-2} + P_{m-1,n} - P_{m-1,n-1} \\ &+ P_{m-1,n-1} - P_{m-1,n-2} \,, \end{aligned} \tag{6}$$

or

$$P_{m,n} = 2P_{m,n-1} - P_{m,n-2} + P_{m-1,n} - P_{m-1,n-2} \qquad m \ge 2,\ n \ge 3 \,, \tag{7}$$

as promised by equation (1).

The boundary values for the sequence $P_{m,n}$ are now determined. When $m = 1$ there is only one row. The path may begin in any column p and end in any column q, where $1 \le p \le q \le n$. For each value of p there are $n - p + 1$ possible values of q; hence, $P_{1,n} = n + (n-1) + \ldots + 2 + 1 = n(n+1)/2$. When $n = 1$ there is only one column and hence there is only one path. Therefore, $P_{m,1} = 1$. When $n = 2$, there are three possible types of path, as shown in Figure 2.

(a) There are two paths consisting entirely of vertical moves.
(b) There are paths containing one horizontal move. This may be in any one of the m rows, so there are m such paths.
(c) There are paths containing one diagonal move. This may be from any row k, $k = 1, \ldots, m-1$, so there are $m - 1$ such paths.

The sum of these three possibilities gives $P_{m,2} = 2 + m + m - 1 = 2m + 1$.

These boundary values and recurrence relation give us an easy method for calculating $P_{m,n}$ for small m and n. These are shown in Table 1. The first row consists of the triangular numbers and the second row is the sequence of square pyramidal numbers, that is, $P_{2,n} = 1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$. These two rows are listed in The On-Line Encyclopedia of Integer Sequences [1], as are the next two. However, the formulae for these do not throw much light on our quest to approximate the order of magnitude of P for large m and n. It suffices to say that

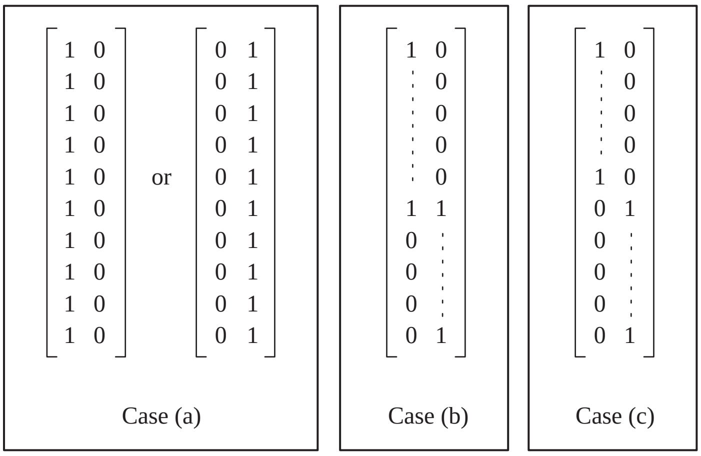

Figure 2. Calculating $P_{m,2}$ as the sum of the three different cases (a), (b) and (c).

Table 1. Values of $P_{m,n}$ for small m and n.

		n = 1	2	3	4	5	6	7	8	9	10
	1	1	3	6	10	15	21	28	36	45	55
	2	1	5	14	30	55	91	140	204	285	385
	3	1	7	26	70	155	301	532	876	1365	2035
	4	1	9	42	138	363	819	1652	3060	5301	8701
m	5	1	11	62	242	743	1925	4396	9108	17469	31471
	6	1	13	86	390	1375	4043	10364	23868	50445	99385
	7	1	15	114	590	2355	7773	22180	56412	130725	280555
	8	1	17	146	850	3795	13923	43876	122468	309605	720885
	9	1	19	182	1178	5823	23541	81340	247684	679757	1710247
	10	1	21	222	1582	8583	37947	142828	471852	1399293	3789297

we do not want to list and analyse all of the nearly four million train paths for $m = n = 10$.

We now solve the recurrence relation and boundary conditions for P. We do this via the generating function for sequence a. Note that a has boundary values $a_{m,1} = P_{m,1} = 1$ and $a_{1,n} = n$.

We rewrite (5) so that the smallest subscripts are m and n, and rearrange for $a_{m,n}$ to give

$$a_{m,n} = a_{m+1,n+1} - a_{m+1,n} - a_{m,n+1} \qquad m \geq 1, n \geq 1\,. \tag{8}$$

Let $G(x,z)$ be the generating function for $a_{m,n}$ defined by

$$G(x,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}x^m z^n\,. \tag{9}$$

It follows from (8) that:

$$\begin{aligned} G(x,z) &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_{m,n}x^{m-1}z^{n-1} - \sum_{m=2}^{\infty}\sum_{n=1}^{\infty} a_{m,n}x^{m-1}z^{n} - \sum_{m=1}^{\infty}\sum_{n=2}^{\infty} a_{m,n}x^{m}z^{n-1} \\ &= G(x,z)\left[\frac{1}{xz}-\frac{1}{x}-\frac{1}{z}\right] - 1 - \sum_{m=2}^{\infty} x^{m-1} - \sum_{n=2}^{\infty} nz^{n-1} + \sum_{n=1}^{\infty} nz^{n} + \sum_{m=1}^{\infty} x^{m} \\ &= G(x,z)\left[\frac{1}{xz}-\frac{1}{x}-\frac{1}{z}\right] + \frac{1}{z-1}, \qquad \text{where } |z|<1\,. \end{aligned}$$

Therefore,

$$\begin{aligned} G(x,z) &= \frac{xz}{(z-1)(xz+x+z-1)} \\ &= \frac{xz}{(1-z)(1-x-z-xz)} \\ &= xz\left[\sum_{j=0}^{\infty} z^{j}\right]\left[\sum_{j=0}^{\infty}(x+z+xz)^{j}\right] \end{aligned} \tag{10}$$

where $|z|<1$ and $|x+z+xz|<1$.

By finding the coefficient of $x^m z^n$, $m \geq 1, n \geq 1$, in the generating function given by the right-hand side of (10), we can determine the corresponding member $a_{m,n}$ of the sequence given by the recurrence relation in (5). To find the coefficient of $x^m z^n$ in $G(x,z)$ we look at the contribution from each of the three terms in the product.

The xz term contributes and has a coefficient of 1, so we are left with selecting $x^{m-1}z^{n-1}$ from $[\sum_{j=0}^{\infty} z^j][\sum_{j=0}^{\infty}(x+z+xz)^j]$.

The contribution from $\sum_{j=0}^{\infty} z^j$ is z^i, where $i \in \{0,\ldots,n-1\}$, which also has a coefficient of 1. It now remains to determine the coefficient of $x^{m-1}z^{n-1-i}$ in $\sum_{j=0}^{\infty}(x+z+xz)^j$, and then to sum over permissible values of i.

We must determine which values of j in $\sum_{j=0}^{\infty}(x+z+xz)^j$ give rise to the term $x^{m-1}z^{n-1-i}$. Consider the term $(x+z+xz)^j$ for an arbitrary j. From this product of j copies of $x+z+xz$, we could select

$$\begin{aligned} xz \text{ from } & k = (m-1)+(n-1-i)-j, \\ x \text{ from } & (m-1)-k = j-(n-1-i), \\ z \text{ from } & (n-1-i)-k = j-(m-1). \end{aligned}$$

The reader can easily verify that the resultant term has x to the power $m-1$ and z to the power $n-1-i$, both as required, and that the total number of selections is j, also as required. We can immediately observe that $\max\{n-1-i, m-1\} \leq j \leq (m-1)+(n-1-i)$.

The relevant coefficient for this arbitrary value of j is

$$\frac{j!}{(j-(n-1-i))!\,(j-(m-1))!\,((m-1)+(n-1-i)-j)!}\,. \tag{11}$$

Hence, the coefficient $a_{m,n}$ of $x^m z^n$ in $G(x,z)$ is

$$a_{m,n} = \sum_{i=0}^{n-1} \sum_{j=\max\{m-1,n-1-i\}}^{(m-1)+(n-1-i)} \frac{j!}{(j-(n-1-i))!\,(j-(m-1))!\,((m-1)+(n-1-i)-j)!}$$

for $m, n \geq 1$.

Now, from equation (4),

$$\begin{aligned} P_{m,n} &= \sum_{p=1}^{n} a_{m,n-p+1} \\ &= \sum_{p=1}^{n} \sum_{i=0}^{n-p} \sum_{j=\max\{m-1,n-p-i\}}^{(m-1)+(n-p-i)} \frac{j!}{(j-(n-p-i))!\,(j-(m-1))!\,((m-1)+(n-p-i)-j)!}. \end{aligned} \tag{12}$$

Having found this expression for $P_{m,n}$ we can interpret it combinatorially. Consider an arbitrary term in the sum in the right-hand side of (12). The arguments of the factorials in the denominator sum to j, and the numerator is $j!$, so it is a trinomial coefficient. Specifically, it is the number of different arrangements of a set of j objects that contains $j-(n-p-i)$ identical objects of type 1, $j-(m-1)$ identical objects of type 2, and $(m-1)+(n-p-i)-j$ identical objects of type 3.

In the context of our problem, these objects are moves that constitute a path from $(1,p)$ to $(m, n-i)$ in the grid. The total number of moves is j. The total number of vertical *steps* to be taken is $m-1$ and the total number of horizontal steps to be taken is $n-p-i$. A horizontal or vertical move is a single step, but the effect of a diagonal move is simultaneous horizontal and vertical steps. So the number of diagonal moves must be $(m-1)+(n-p-i)-j$; these are type 3 objects. Thus the number of vertical moves is $m-1-\{(m-1)+(n-p-i)-j\} = j-(n-p-i)$ (type 1) and of horizontal moves is $n-p-i-\{(m-1)+(n-p-i)-j\} = j-(m-1)$ (type 2).

Clearly, j must be greater than or equal to each of $m-1$ and $n-p-i$, and less than or equal to $(m-1)+(n-p-i)$. Summing over permissible j yields the total number of paths from $(1,p)$ to $(m,n-i)$, and the limits of summation for i and p ensure that (12) counts all the relevant paths in the grid.

References

[1] Sloane, N.J.A. (2006). The On-Line Encyclopedia of Integer Sequences. www.research.att.com/∼njas/sequences/ (accessed 2 March 2008).