Pantographic polygons

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Abstract

Necessary and sufficient conditions are given for a polygon to be pantographic. The property is held by all regular polygons and by many nonregular polygons having an even number of sides, but by no polygons having an odd number of sides. Many processes are shown by which new pantographic polygons can be constructed from old, including a process of adjunction of polygons.

In Euclidean plane geometry, let $\Omega = \mathrm{PQRS}$ be any nondegenerate plane quadrilateral, and let F, G, H, I be the midpoints of the successive sides QR, RS, SP, PQ. Then $\Lambda := \mathrm{FGHI}$ is a parallelogram; for each pair of opposite sides is parallel to a diagonal of Ω.

The construction is reversible: if $\Lambda := \mathrm{FGHI}$ is any parallelogram and P any point in the plane (or even outside the plane), then drawing segments PQ, QR, RS with midpoints I, F, G results in H being the midpoint of SP. Thus the set $\mathcal{Q}(\Lambda)$ of quadrilaterals having the vertices of Λ as the midpoints of their sides (taken in order in both cases) is doubly-infinite. We call $\mathcal{Q}(\Lambda)$ the *pantograph of* Λ, and its members the quadrilaterals *generated by* Λ. The construction was described in some detail in [1] and [2]. Unfortunately there are problems of degeneracy. Even if Λ is not degenerate, Ω may be so, and vice versa.

What other plane polygons besides parallelograms generate polygons in this way?

Definition 1. We shall say that a polygon Λ is *pantographic* if, when all its vertices are taken as the midpoints of the segments of a polygonal arc in order, it causes this polygonal arc to be closed, that is, to be itself a polygon. We then call this a *generated polygon* of Λ. The *order* of any polygon is the number of its sides.

If the construction of a generated polygon starts by using initially a point P and a vertex V of Λ (that is, if the first drawn segment PQ say has midpoint V), call it a $[\mathrm{P},\mathrm{V}]$-*construction*. Note that P and V determine the generated polygon uniquely. Note also that if for all possible P a $[\mathrm{P},\mathrm{V}]$-construction results in a polygon, then the same is true of $[\mathrm{P},\mathrm{W}]$-constructions for each other vertex W of Λ. So in testing to see if a polygon is pantographic, any vertex may be chosen as first vertex. Note our verbal distinction between a *polygonal arc* $\mathrm{L} = \mathrm{A}_1\mathrm{A}_2\ldots\mathrm{A}_m$, from A_1 to A_m, and a *polygon* $\Lambda = \mathrm{A}_1\mathrm{A}_2\ldots\mathrm{A}_m$, having a last side $\mathrm{A}_m\mathrm{A}_1$. We shall show that the property of being pantographic is possessed by all regular $2n$-gons, and by many

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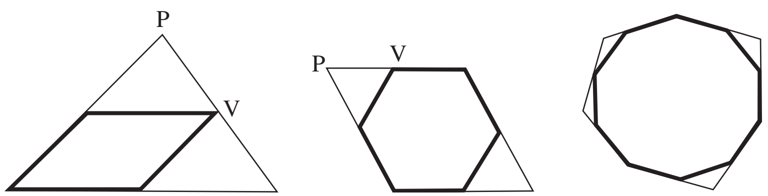
non-regular $2n$-gons as well, but by no polygons of odd order. We give several methods of finding new pantographic polygons from old.

In this paper we do not assume that the polygon Λ is convex or non-selfintersecting, but (as a matter of convenience rather than necessity) we do assume initially that Λ is *nondegenerate*, that is, *no three vertices are collinear*, so that all vertices are distinct and all sides have positive length. On the other hand the generated polygons may be degenerate, through failure of one or more of these conditions. There are many forms of degeneracy which can arise when the initial point P is ill chosen, resulting in a count of the apparent number of vertices or sides being less than the order of Λ, by some sides having length zero, or by pairs of non-consecutive vertices coinciding, or some vertices being collinear. See Figure 1.



$2n = 4,\ k = 3$ $2n = 6,\ k = 4$ $2n = 10,\ k = 7$

Figure 1. Examples of regular $2n-$gons with degenerate generated polygons of k apparent sides.

It can be proved that: *For any* $n > 2$ *and any* k *satisfying* $2n - [2n/3] \leq k \leq 2n - 1$ *there exists a pantographic* $2n$*-gon with a generated polygon having* k *apparent sides, that is, having* $2n - k$ *sides of zero length.*

For nondegenerate cases the order of any generated polygon equals the order of Λ. Any degenerate case is the limit of a sequence of nondegenerate cases: for a rigorous discussion of this for quadrilaterals, see [1].

As we shall see, for some given polygons M it can happen that a generated polygon arises by this construction for certain choices of starting pair P, V but not for others: we do not call these polygons M pantographic. But we can still use the notation $\mathcal{Q}(\mathrm{M})$ for the set of generated polygons, which is possibly empty. For example, $\mathcal{Q}(\mathrm{M})$ is empty when M is a trapezium which is not a parallelogram. We show in fact that the cardinality of $\mathcal{Q}(\mathrm{M})$ can only be 0, 1 or infinity.

We start by formulating an analytic version of Definition 1 which is less hampered by considerations of degeneracy, and which leads to necessary and sufficient conditions and easy proofs of ensuing results. Let an arbitrary origin O be taken in the plane and for any point P write **P** for its position vector OP.

Definition 2. Let $\Lambda = \mathrm{A}_1\mathrm{A}_2\ldots\mathrm{A}_m$ be a nondegenerate polygon of m vertices A_j and sides $\mathrm{A}_j\mathrm{A}_{j+1}(\mathrm{A}_{m+1} = \mathrm{A}_1)$, and let $\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_m$ be a polygonal arc such that

$$\mathbf{A}_j = \tfrac{1}{2}(\mathbf{X}_j + \mathbf{X}_{j+1}) \qquad (j = 1, 2, \ldots, m-1)\,.$$

If also $\mathbf{A}_m = \frac{1}{2}(\mathbf{X}_m + \mathbf{X}_1)$ then the polygon $\Omega = \mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_m$ is said to be *generated by* Λ. If Ω is a generated polygon for every choice of point $\mathbf{X}_1$ in the plane, then Λ is said to be *pantographic.*

The definition is independent of the choice of origin O. The earlier remarks show that a quadrilateral is pantographic if and only if it is a nondegenerate parallelogram.

Theorem 1. *No polygon Λ of odd order is pantographic. Every polygon of odd order ≥ 3 has a unique generated polygon.*

Proof. Let Λ be as above. The system of equations

$$\mathbf{A}_j = \tfrac{1}{2}(\mathbf{X}_j + \mathbf{X}_{j+1}) \quad (j = 1, 2, \ldots, m), \qquad \mathbf{X}_{m+1} = \mathbf{X}_1 \tag{1}$$

has to be solved. Let m be odd. Adding all equations gives

$$\mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_m = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_m\,. \tag{2}$$

Adding the odd-numbered equations gives

$$2(\mathbf{A}_1 + \mathbf{A}_3 + \cdots + \mathbf{A}_m) = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_m + \mathbf{X}_1. \tag{3}$$

Therefore $\mathbf{X}_1$ has a determined value, $\mathbf{X}_1 = \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 - \cdots + \mathbf{A}_m$. Writing $\mathbf{X}_{j+1} = 2\mathbf{A}_j - \mathbf{X}_j$ to find successive $\mathbf{X}_j$, we end up with

$$\begin{aligned}\mathbf{X}_m &= 2(\mathbf{A}_{m-1} - \mathbf{A}_{m-2} + \mathbf{A}_{m-3} - \cdots + (-1)^m\mathbf{A}_1) + \mathbf{X}_1\\ &= -\,2(\mathbf{X}_1 - \mathbf{A}_m) + \mathbf{X}_1\,,\end{aligned}$$

that is, $\mathbf{A}_m = \frac{1}{2}(\mathbf{X}_m + \mathbf{X}_1)$; thus $\Omega := \mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_m$ is a generated polygon of Λ, and clearly it is the unique solution of (1).

Corollary 1. *If Λ is a regular polygon of odd order, the unique generated polygon is the regular polygon circumscribing Λ symmetrically.*

Theorem 2. *A nondegenerate polygon of even order $\Lambda = \mathrm{A}_1\mathrm{A}_2\ldots\mathrm{A}_{2k}$ is pantographic if and only if*

$$\mathbf{A}_1 + \mathbf{A}_3 + \mathbf{A}_5 + \cdots + \mathbf{A}_{2k-1} = \mathbf{A}_2 + \mathbf{A}_4 + \mathbf{A}_6 \cdots + \mathbf{A}_{2k}\,. \tag{4}$$

Proof. Assume that Λ is pantographic and that $\Omega = \mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_{2k}$ is any generated polygon. Then (1) holds with $m = 2k$, and hence (2) holds. In place of (3) we get (4) by a similar calculation, but $\mathbf{X}_1$ can be arbitrary.

Conversely, assume (4). Then it is easily verified that for any X_1 the points defined by

$$\mathbf{X}_j = 2(-1)^j[\mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 - \cdots + (-1)^j\mathbf{A}_{j-1}] + (-1)^{j-1}\mathbf{X}_1 \qquad (j = 2, 3, \ldots, 2k) \tag{5}$$

are the vertices of a polygon generated by Λ, since the $\mathbf{X}_j$ satisfy equations (1). Because a polygon eventuates for every choice of X_1, Λ is pantographic.

Corollary 2. *There exist nonpantographic polygons of all orders ≥ 3, and pantographic polygons of all even orders ≥ 4.*

Corollary 3. *If* $\Lambda = A_1A_2 \ldots A_{2k}$ *is pantographic, then so is*

$$\Lambda = A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)} \ldots A_{\sigma(2k)}\,,$$

where σ *is any permutation of* $(1, 2, \ldots, 2k)$ *which restricts to a permutation on each of* $(1, 3, 5, \ldots, 2k-1)$ *and* $(2, 4, 6, \ldots, 2k)$.

Let $\mathcal{O}$ and $\mathcal{E}$ denote the sets of odd-numbered and even-numbered vertices of Λ respectively, and attribute mass 1 to each vertex. Then (4) has the natural interpretation: *point-sets* $\mathcal{O}$ *and* $\mathcal{E}$ *have the same centre of mass.* So:

Corollary 4. *If* $\Lambda = A_1A_2 \ldots A_{2k}$ *is pantographic, then so is any nondegenerate polygon obtained by applying any transformation of the plane which preserves the centre of mass of both sets, in particular by applying to one or the other of* $\mathcal{O}$, $\mathcal{E}$ *any dilation or rotation about the common centre of mass, leaving the other set unmoved.*

Corollary 5. *If* $\Lambda = A_1A_2 \ldots A_{2r}$ *and* $M = B_1B_2 \ldots B_{2s}$ *are pantographic* $2r$*-gon and* $2s$*-gon respectively, then the polygon*

$$N := A_1A_2 \ldots A_{2r}B_1B_2 \ldots B_{2s}$$

is a pantographic $2(r+s)$*-gon, if nondegenerate.*

Proof. The property (4) for each of Λ and M separately gives two equations; their sum is property (4) for N.

Used separately or in conjunction, Corollaries 3, 4 and 5 give many ways of constructing new pantographic polygons from old.

Corollary 6. *A* $2k$*-gon is either pantographic, or has no generated polygon at all.*

Proof. If Λ is not pantographic then (4) does not hold, and no solution of (1) exists.

It follows from Theorem 1 (Corollary 1) and Theorem 2 (Corollary 6) that every polygon has exactly 0, 1, or infinitely many generated polygons.

Corollary 7. *Every regular polygon of even order* $2k$, $k \geq 2$, *is pantographic.*

Proof. It suffices to assume that the vertices of the polygon lie on the unit circle and to use the representation of the complex plane, with vertices $A_r = \exp(ir\pi/k)$ $(r = 1, 2, \ldots, 2k)$, say, and then to verify property (4) in its corresponding form.

Corollary 8. *A nondegenerate hexagon* $\Lambda = A_1A_2 \ldots A_6$ *is pantographic if and only if there exists a point* K *such that* $A_1A_2A_3K$ *and* $KA_4A_5A_6$ *are both parallelograms.*

Proof. Corollary 8 is intuitively obvious: any hexagon $\Omega = X_1X_2 \ldots X_6$ generated by Λ determines a point K as midpoint of X_1X_4 having the stated property. An analytic proof proceeds as follows. Assume that there exists a point K, and take it as origin. Then $\mathbf{A}_1 + \mathbf{A}_3 = \mathbf{A}_2$ and $\mathbf{A}_4 + \mathbf{A}_6 = \mathbf{A}_5$. Therefore $\mathbf{A}_1 + \mathbf{A}_3 + \mathbf{A}_5 =$

$\mathbf{A}_2 + \mathbf{A}_4 + \mathbf{A}_6$, confirming (4). Conversely suppose Λ is pantographic and let K be the fourth vertex of the parallelogram $A_1A_2A_3K$. By a reversal of the previous argument, $KA_4A_5A_6$ is also a parallelogram. See Figure 2.

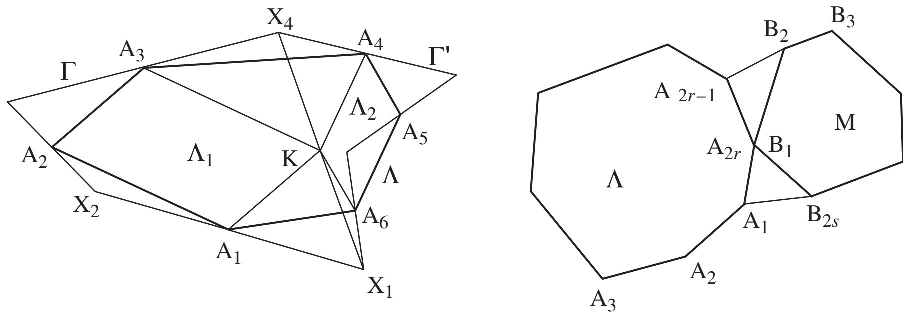


Figure 2. Illustrating Corollaries 8 and 9, and Theorem 4.

Corollary 9. *A polygonal arc* $L = A_1A_2A_3A_4$ *of three segments with no three vertices collinear can be completed to form a pantographic hexagon if and only if* A_1, A_2, A_3, A_4 *are not vertices of a parallelogram, and in this case this can be done in infinitely many ways.*

Proof. The condition is required so as to ensure that there exists a nondegenerate completion. Choose an origin O and two further distinct points A_5 and A_6 not collinear with any of A_1, A_2, A_3, A_4. If the polygon $\Lambda = A_1 \dots A_6$ is pantographic then $\mathbf{A}_1 + \mathbf{A}_3 + \mathbf{A}_5 = \mathbf{A}_2 + \mathbf{A}_4 + \mathbf{A}_6$, and since $\mathbf{A}_5 \neq \mathbf{A}_6$ then $\mathbf{A}_2 - \mathbf{A}_1 \neq \mathbf{A}_3 - \mathbf{A}_4$ so L is not three sides of a parallelogram. The converse is proved similarly.

In order to avoid consideration of degenerate cases, in the remaining paragraphs we relax Definitions 1 and 2 by no longer requiring nondegeneracy as a requirement for a polygon to be pantographic.

Theorem 3. *Any polygonal arc* $L = A_1A_2 \dots A_{2k-1}$ *of* $2k-1$ *segments may be completed to a (possibly degenerate) pantographic polygon by use of a last vertex* A_{2k}*, which is uniquely determined.*

Proof. It suffices to add the vertex $\mathbf{A}_{2k} := \mathbf{A}_1 + \mathbf{A}_3 + \mathbf{A}_5 + \cdots + \mathbf{A}_{2k-1} - (\mathbf{A}_2 + \mathbf{A}_4 + \mathbf{A}_6 + \cdots + \mathbf{A}_{2k-2})$ and form the polygon $\Lambda = A_1A_2 \dots A_{2k}$.

Theorem 4. *Let* $\Lambda = A_1A_2 \dots A_{2r}$ *and* $M = B_1B_2 \dots B_{2s}$ *be pantographic* $2r$*-gon and* $2s$*-gon respectively, with* $A_{2r} = B_1$*. Then the polygon*

$$Z := A_1A_2 \dots A_{2r-1}B_2 \dots B_{2s}$$

is a (possibly degenerate) pantographic $2(r+s-1)$*-gon.*

Given Λ and M as in the theorem, and nondegenerate, a rotation of M around B can always be found ensuring that Z is nondegenerate, since only finitely many angles of rotation need to be avoided; therefore nondegenerate pantographic polygons can be constructed with great freedom by the process of adjoining two polygons with a common vertex.

Conversely, given a pantographic $2k$-gon we can, by a reversal of this process, remove from it three adjacent vertices and add the fourth vertex of the parallelogram, to produce a pantographic $2(k-1)$-gon.

A final remark: Definitions 1 and 2 do not actually require that they be interpretated in the plane; it suffices that the points inhabit a Euclidean space E^{m-1}. With this understanding, the previous results suitably reinterpreted still hold. If Λ is a pantographic $2k$-gon, (4) shows that it lies in E^{2k-2}; for example a parallelogram lies in a plane but its generated quadrilaterals can lie in E^3.

References

[1] Miller, J.B. (2007). Plane quadrilaterals. *Gaz. Aust. Math. Soc.* **34,** 103–111.
[2] Miller, J.B. (2008). Pantographs and cyclicity. *Gaz. Aust. Math. Soc.* **35,** 35–42.