On the definition of topology — a cautionary tale

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Elastic
It might just be a set
But you'd best not forget
That the consequence could be quite drastic

Should you carelessly choose
And then heedlessly use
Definitions too vague and elastic

Anon.

Introduction

The traditional definition of a topological space is as follows:

Definition 1. A topology on a set X is a collection of subsets of X, called open sets, for which:

(1) the intersection of any finite collection of open sets is an open set,
(2) the union of any collection of open sets is an open set.

The aim of this paper is to highlight an important subtlety in this definition, which is usually glossed over. In order to emphasise our point, we prove a possibly contentious result! Recall that the Euclidean topology on the set $\mathbb{R}$ of real numbers is the topology generated by the base of open intervals. We have:

Claim. *If one is too liberal with one's interpretations of the terms involved, it is not possible to prove that the Euclidean 'topology' on* $\mathbb{R}$ *is a topology in the above sense.*

It will become clearer what we mean by this claim later in the paper. But let us cut to the chase: the contentious issue in the above definition is the distinction between sets and collections. The definition is careful with its words: on a *set* X, a topology is a *collection* of subsets of X. However, a collection is a notion that has no universally accepted definition. Indeed, it is a word/notion that is very widely employed in set theory texts, but seldom explicitly defined. In many cases, it is simply a 'façon de parler'. An interesting discussion of collections is given by

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Potter [23, Chapter 1], who writes that since Zermelo's 1904 enunciation of the axiom of choice,

> ... the dominant view among mathematicians has been that collections exist in extension quite independent of the language we have at our disposal for defining them ... Potter [23, p. 13]

Set theories vary from author to author, but it is widely recognised that there are general objects called classes, and more restrictive things called sets (and classes that aren't sets are called proper classes). The word 'collection' is commonly used to mean something at least as general as class, and it is in this rather vague sense that we use it in this paper. We don't explicitly define the term collection; indeed, it was not our purpose to give explicit meaning to Definition 1, but rather to highlight the inherent problem.

Collections, classes, sets or families?

There is considerable variation in topology texts on the definition of a topological space. Many texts give definitions, like the one we have given above, without any comment on the meaning of the terms collection, class etc. In some texts, like Bourbaki [2], Thron [26], Jänich [13], topologies are defined as sets of subsets; these texts entirely avoid the issues raised in this paper. Some authors use the words family, or class, instead of collection. In many books (e.g. Dugundji [5], Engelking [6]), a family is understood to be an indexed set; so this situation is effectively the same as the Bourbaki definition. Also, in texts like Hu [12], the words families and collections are just synonyms for sets, while a class is a distinct concept. This usage, designed to 'avoid terminological monotony' [10, p. 1], is the position taken by Halmos' in *Naive Set Theory*, which is a common set theory reference in topology books. In some texts, like Moore [18] and Simmons [25], the words classes, families and collections are all effectively synonyms for sets. In Kelley's influential text [15, p. 1], set, class, family and collection are declared to be synonymous, but Kelley adds a footnote saying that 'this statement is not strictly accurate'; he then distinguishes between sets and classes, without clarifying the role of families and collections. In Fuks and Rokhlin [8], the definition of a topological space is that of a *class* of subsets, and the impression that is given is that classes are more general than sets. Overall, the general impression is that authors of topology texts have shown little enthusiasm for the distinction between the concepts of set, class and collection. It is remarkable that many texts give chapters or appendices on set theory and yet pass in silence over their choice of terminology in their definition of a topological space.

Let us say a few words on the distinction between classes and sets. The notion of a class appears in different guises in different set theories. It is a fundamental concept in the von Neumann–Bernays–Gödel set theory (see [22]), which is used by Dugundji [5]. Classes didn't appear at all in traditional treatments of ZFC (Zermelo–Fraenkel + Choice), but classes can certainly coexist happily with sets in ZFC (see for example [19]). According to [16, p. 9], 'a general specifiable collection, which may or may not be a set, will be called a class'; in other words, a class is a collection defined by a formula. Depending on one's precise definition of 'class', in ZFC, it follows from the separation axiom that every subclass of a set is a set [19, Chapter 3]. In this case, if a topology on a set X is defined as a class of subsets, then it is a subclass of the power set $P(X)$ of X, and so the topology is a set. So with appropriate definitions, it makes little difference if one uses sets of subsets, like Bourbaki, or classes of subsets, like Fuks–Rokhlin. For more on classes and sets, see [20].

We should mention that these days it is not uncommon to extend the definition of a topological space to the case where X itself is a proper class; see [16, p. 202], [3]. Commonly occurring examples of such 'spaces' are the class **On** of all ordinals, with the order topology, or the class **No** of all surreal numbers, again with the order topology [4]. Notice that in such cases, the topology on X can not be a class (let alone a set), since the entire space X is a member of the topology, and if X is a proper class, it cannot belong to a class. This more or less forces one to conceive of collections in a broader context than classes. Another such context is category theory; see [17]. One place where a collection is explicitly defined is [23]; here $\mathcal{C}$ is a collection if $\mathcal{C} = \{x \colon x \in \mathcal{C}\}$.

Returning to Definition 1, notice that collections are used twice: firstly, a topology is a *collection* of subsets and secondly, the union of a *collection* of open sets is an open set. The first aspect raises an obvious question: is there an example of a topology (on a set X) that isn't a set? We are not aware of such an example in the literature: we exhibit an example, in a somewhat different context, at the end of this paper. It is the second aspect that underlies our Claim. Even if one has a clear notion of 'collection', the idea of taking a union of an arbitrary collection of sets can be problematic. It is clearly defined by [23], but elsewhere it is regarded as being dangerous; see [7]. In our context of topological spaces, the problem is: how does one know that a union of a collection of open sets is a set? The point of this paper is to observe that indeed, such a union may not be a set. Is such a caution really necessary? We believe that it might have some merit. It seems that the force of tradition is so strong that even in some set theory books, topological spaces are defined as in Definition 1. Thus, for example, the 'collection of all sets' [11, p. 153] may be considered, while in the same book, the union of 'arbitrary collections' of open sets [11, p. 122] may also be entertained.

Which set theory?

ZFC — Finger Lickin' Good
Long 'ere Sanders and his secret spices
Reinvented the flavour of chook
Herrs Zermelo and Frankel's devices
Near rewrote the old set theory book.
Anon.

The reader may expect that our Claim relies on technical aspects of set theory. In fact, the proof is very simple and easy to follow, provided the reader is willing to accept a few basic facts. Actually, the set theoretic framework we adopt is that of nonstandard analysis. The particular version we use is Nelson's *internal set* *theory*; a good introductory reference is Robert [24]. It is defined by adding three axioms to ZFC; we denote it ZFC+IST. For the notion of collection, we adopt any definition (e.g. Potter [23]) which allows us to speak of the collection of all standard natural numbers and which permits unions of collections of sets. We remark that the expression 'collection of all standard natural numbers' is quite common usage; for example, see the preface to [14]. Later in this paper we prove:

Theorem 1. *In ZFC+IST, the Euclidean 'topology' on* $\mathbb{R}$ *is not a topology in the sense of Definition 1.*

We remark that internal set theory is a conservative extension of ZFC; that is, any sentence expressible in ZFC and provable in ZFC+IST is already provable in ZFC. In particular, any contradiction in ZFC+IST is already present in ZFC [21]. So our Theorem gives a model of ZFC for which the Euclidean 'topology' on $\mathbb{R}$ fails Definition 1.

We won't go through the three IST axioms (Idealisation, Standardisation and Transfer), but let us outline the facts that we need for this paper. The basic idea is that a predicate standard is added to ZFC set theory; so every set (and in particular, every number) is either standard, or nonstandard. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are standard, as are the numbers $0, 1, 2, 3$. The transfer axiom says that if a set E is defined by a formula all of whose parameters are standard, then E is standard. So, for example, if a and b are standard integers, the set $\{x \in \mathbb{Z}\colon a \leq x \leq b\}$ is standard. But the elements of a standard set aren't necessarily standard. In fact, one has:

Lemma 1. *In* ZFC+IST*, all the elements of a set E are standard if and only if E is standard and finite* [24, Theorem 2.4.2.].

It follows in particular that every infinite set contains a nonstandard element.

The upshot of the axioms is that inside the set of integers $\mathbb{Z}$, there is a collection $\mathcal{S}$ of *standard* numbers which has the following remarkable quartet of properties:

(a) $0 \in \mathcal{S}$ and $1 \in \mathcal{S}$,
(b) for all $x \in \mathcal{S}$, one has $-x \in \mathcal{S}$,
(c) for all $x, y \in \mathcal{S}$, one has $x + y \in \mathcal{S}$ and $xy \in \mathcal{S}$,
(d) there exists $l \in \mathbb{N}$ such that $|x| < l$, for all $x \in \mathcal{S}$.

If $\mathcal{S}$ was a set, the first three conditions would say that $\mathcal{S}$ is a subring with unity of $\mathbb{Z}$ (and hence, equal to $\mathbb{Z}$). But $\mathcal{S}$ is not a set. One can say that, from property (d), the collection $\mathcal{S}$ is contained in a finite set. What are these standard numbers? Any concrete number that one could write down would be standard. In some sense, the standard numbers are the observable or accessible numbers. There is no maximal standard number, and yet they are limited, by some (nonstandard) limit l. Conversely, there are infinitely many nonstandard numbers, but one can't explicitly write down a single one. These nonstandard concepts take a little getting used to; for example, the set $\{x \in \mathbb{Z}\colon |x| < l\}$ is a finite (but nonstandard) set that contains a subcollection $\mathcal{S}$ that is not a set, which at first seems rather alarming. The genius of the IST system is that it will never lead one to a contradiction, that one wouldn't have encountered in ZFC anyway [21].

We remark that there is a version of internal set theory which is built on von Neumann–Bernays–Gödel set theory, rather than on ZFC; see [1]. In this theory, $\mathcal{S}$ is a class.

Proof of the theorem

What we actually prove is that the discrete 'topology' on $\mathbb{Z}$ is not a topology in the sense of Definition 1. This suffices since, if the Euclidean 'topology' on $\mathbb{R}$ was a topology, the induced discrete 'topology' on $\mathbb{Z}$ would also be a topology.

In the discrete 'topology' on $\mathbb{Z}$, each singleton set is open. In particular, for each standard number $x \in \mathcal{S}$, the singleton set $\{x\}$ is an open set. But the union of the collection of the sets $\{x\}$, for $x \in \mathcal{S}$, is nothing other than $\mathcal{S}$, which is not a set. So here we have a union of a collection of open sets which is not a set. Thus, the second condition in Definition 1 is not satisfied.

We remark that in a similar way, it is not difficult to show that, using Definition 1 and the above lemma, every Hausdorff space with standard open sets is finite.

A new definition

It would seem that some authors, faced with the distinction between sets and classes, have chosen to use 'collections', thinking perhaps that this is a more prudent approach. The resulting problem can be removed of course, if one adopts the Bourbaki approach of using sets rather than collections (or by making 'collection' synonymous with 'set'). Notice however that the problem only occurred with the second occurrence of 'collection' in Definition 1. For people who want to continue to work with topologies of open sets but who see no reason to restrict the topology to be a set itself, one possibility is to adopt the following:

Definition 2. A topology on a set X is a collection of subsets of X, called open sets, for which:

(1) the intersection of any finite set of open sets is an open set,
(2) the union of any set of open sets is an open set.

This is the definition adopted in [19, p. 45] and [9], although 'collection' isn't defined in these references. An alternate approach is to define a topology as a collection of subcollections, and allow unions over arbitrary collections; this is the approach taken in [23, Chapter 9]. Definition 2 raises the obvious question: is there a topology that isn't a set? This is the topic of the next section.

A topology, on a set, that isn't a set

Consider the set $\mathbb{Z}$ and the collection $\mathcal{C}$ which consists of: the empty set $\varnothing$, the set $\mathbb{Z}$, and the sets of the form $U_x = \{y \in \mathbb{Z} \colon |y| \leq x\}$, for $x \in \mathcal{S}$. The collection $\mathcal{C}$ is not a set, since $\mathcal{S}$ is not set. We claim that $\mathcal{C}$ is a topology, in the sense of Definition 2, and anticipating this, we say that the members of $\mathcal{C}$ are *open sets*. Clearly, $\mathcal{C}$ is closed under finite intersections and under finite unions of open sets.

It remains to show that $\mathcal{C}$ is closed under the union of an arbitrary set of open sets. It suffices to consider a set A which consists entirely of open sets of the form U_x, $x \in \mathcal{S}$. Each of the sets U_x is a standard set, and hence A is a set all of whose elements are standard. So A is finite, by the lemma. Thus the union of the elements of A is also an open set, as we required.

We remark that the same kind of example can be obtained by starting with almost any familiar topological space and restricting to the standard open sets; the Euclidean topology for example.

Making sense out of nonsets

The key to the above arguments has been the existence of a collection of numbers that isn't a set. We used internal set theory as a convenient vehicle for finding a model of ZFC in which we can name a collection of natural numbers that is not itself a set in the model. This seemed like a sensible approach since many people have some familiarity with nonstandard analysis, but it is by no means the only way to proceed. To complete this article we wish to give an alternative construction of a model of ZFC in which we can name nonset collections of natural numbers. The validity of the approach depends only on the Compactness Theorem for elementary logic — encountered in any first course on model theory — in contrast with the comparatively high-powered model theory required to prove that ZFC+IST is a conservative extension of ZFC. The following method also serves to demonstrate that our central examples are not so much a quirk of nonstandard analysis, but a quirk of logic.

First recall that the ZFC axioms enable the definability of a unique[1] set $\varnothing$ with no members and a unique smallest set ω satisfying:

(1) $\varnothing \,\epsilon\, \omega$;
(2) $X \,\epsilon\, \omega \Rightarrow X \cup \{X\} \,\epsilon\, \omega$.

Of course ω is nothing other than the nonnegative integers, or more precisely, the value of ω in any model of set theory is the interpretation of the nonnegative integers in that model. In our familiar world of set theory (whatever that is for you!) ω is often given the notation $\mathbb{N}_0$, and the relation ϵ is often denoted by $\in$. The numbers $0, 1, 2, \ldots$ are the usual names for $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots$.

Now for the construction. Augment the language of ZFC by adjoining a new constant symbol c, and consider the following set Σ of sentences:

$$\mathrm{ZFC} \cup \{c \,\epsilon\, \omega\} \cup \{i \,\epsilon\, c \mid i \in \mathbb{N}_0\}.$$

Any finite subset of Σ has a model: simply interpret c as a sufficiently large natural number in our familiar model of set theory. The Compactness Theorem now ensures that Σ has a model, say $\langle M; \epsilon, c\rangle$. Since Σ includes the ZFC axioms, $\langle M; \epsilon\rangle$

[1] This is analogous to the ability of the usual group axioms in the language $\{\cdot, ^{-1}\}$ to define a unique multiplicative identity element in any model of group theory. We can unambiguously use $\varnothing$ and ω in sentences as an abbreviation of more complicated formulæ in the same way that one could use 1 in group theory (restricted to the language $\{\cdot, ^{-1}\}$) as an abbreviation of $x^{-1} \cdot x$.

is a model of set theory. In this model, c is a natural number (it is an element of ω). Consider the collection $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, whose members are (some of the) elements of c in the model $\langle M; \epsilon\rangle$. Clearly $\mathbb{N}_0$ satisfies both properties (1) and (2), but $\mathbb{N}_0$ doesn't equal the unique set ω possessing properties (1) and (2); indeed, the elements of $\mathbb{N}_0$ are all contained in a proper subset c of the set ω. Hence $\mathbb{N}_0$ is the desired nonset collection.

To provide some clarification, here is a different description of what we have done here. In the model $\langle M; \epsilon\rangle$, as in any model, the sets, or more precisely M-sets, are just the elements of M. There is a correspondence between M-sets of M and subsets of M: to each $x \in M$, we can associate the subset $\bar{x}$ of M, consisting of all elements related by ϵ to x in $\langle M; \epsilon\rangle$. So the M-sets correspond to some of the subsets of M. This correspondence is not surjective. For example, the set M itself does not correspond to any element of itself — within the model $\langle M; \epsilon\rangle$ it is the proper class of all M-sets. Similarly, our argument identified $\mathbb{N}_0$ as a subset of M not equal to $\bar{x}$ for any $x \in M$, even though it is a subset of $\bar{\omega}$.

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