Puzzle corner

Norman Do[*]

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 8 is 1 September 2008. The solutions to Puzzle Corner 8 will appear in Puzzle Corner 10 in the November 2008 issue of the *Gazette*.

Watchful wombats

An odd number of wombats are standing in a field so that their pairwise distances are distinct. If each wombat is watching the closest other wombat to them, show that there is at least one wombat who is not being watched.

Photo: Xavier Lukins

Rubik's cube

Rubik decides to write a positive integer on each face of a cube. Then to each vertex, he assigns the product of the numbers written on the three faces which meet at that vertex. Rubik then notes that the sum of the numbers assigned to the vertices is 2008. What are the possible values for the sum of the numbers that Rubik wrote on the faces of the cube?

[*]Department of Mathematics and Statistics, The University of Melbourne, VIC 3010.
E-mail: N.Do@ms.unimelb.edu.au

Crowded subsets

Let $S = \{1, 2, \ldots, n\}$ and call $A \subseteq S$ *crowded* if it has the following three properties:

- there are at least two elements in the set A;
- the elements of A form an arithmetic progression;
- the arithmetic progression cannot be extended in either direction using elements of S.

How many crowded subsets of S are there?

Photo: Andy Kelly

Clock confusion

A faulty clock has hour and minute hands which are indistinguishable. In the 12-hour period between noon and midnight, how many moments are there when it is not possible to tell the time on this clock?

Give and Take

Two people, Give and Take, divide a pile of one hundred coins between themselves as follows. Give chooses a handful of coins from the pile and Take decides who will get them. This is repeated until all coins have been taken or until one of them has taken nine handfuls. In the latter case, the other person is allowed to take all of the remaining coins in the pile. What is the largest number of coins that Give can be sure of getting?

Playing with polynomials

(1) The quadratic polynomial $x^2 + 100x + 1$ is written on the blackboard. People enter the room one by one and change the polynomial by adding or subtracting 1 to the x coefficient or the constant coefficient. At some later stage, the polynomial has been changed to $x^2 + x + 100$. Was there necessarily a point in time when the polynomial written on the blackboard had integer roots?

(2) I am thinking of a polynomial $P(x)$ with non-negative integer coefficients. If you give me a number a and ask for the value of $P(a)$, then I will tell you the answer. How many questions of this sort do you need to ask in order to determine $P(x)$?

(3) Find a polynomial $F(x, y)$ in two variables such that, for every non-negative integer n, the equation $F(x, y) = n$ has exactly one solution in non-negative integers.

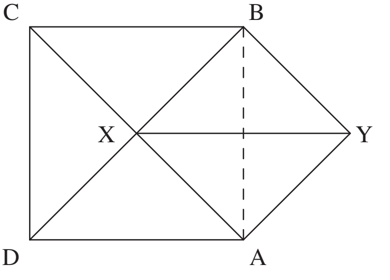
Solutions to Puzzle Corner 6

The \$50 book voucher for the best submission to Puzzle Corner 6 is awarded to Ian Wanless.

Pasting pyramids

Warren Brisley: Let the square pyramid be $ABCDX$, where X is the apex, and let the tetrahedron be $ABXY$. The diagram shows the resulting solid, viewed from above the apex of the square pyramid.

Note that $\overrightarrow{XY} = \overrightarrow{DA}$, so X and Y are at the same height above the base of the square pyramid. Since $|DX| = |AY|$, one can deduce that $\overrightarrow{DX} = \overrightarrow{AY}$. Therefore, the points A, D, X, Y are coplanar and, by a similar argument, the points B, C, X, Y are coplanar. So the resulting solid has five faces — two parallelograms ($ADXY$ and $BCXY$), two triangles (CDX and ABY) and one square ($ABCD$).



Million dollar question

Allison Plant: Let a represent the number of \$1 notes, b the number of \$10 notes, c the number of \$100 notes and d the number of \$1000 notes. The conditions of the problem imply that $a+b+c+d = 500\,000$ and $a+10b+100c+1000d = 1\,000\,000$. Subtracting one equation from the other, we obtain

$$9b + 99b + 999c = 500\,000.$$

This is impossible, since the left-hand side is divisible by 9 while the right hand side is not. So one cannot have exactly half a million notes with a total value of exactly one million dollars.

Collecting coins

Barry Cox: Alex can certainly take all of the coins in odd positions or all of the coins in even positions. So he can guarantee to end up with at least half of the money on the table and, hence, at least as much money as Bree.

Symmetric sets

Andriy Olenko: The set S can consist of one point or two points. We will show that if S contains $n \geq 3$ points, then they must be the vertices of a regular n-gon. If the points of S all lie on a line in the order $P_1, P_2, \ldots, P_n$, then the perpendicular bisector of P_1P_2 is not an axis of symmetry for S. Therefore, the points of S do not all lie on a line and there exists a circular disk which contains all points of S, with at least three points of S on its circumference. If the points on the circumference of this disk are not the vertices of a regular polygon, then there are three consecutive points A, B, C such that $AB \neq BC$. In this case, the perpendicular bisector of AC is not an axis of symmetry for S. So all points of S which lie on the circumference of the disk must be the vertices of a regular polygon. If there is an additional point P inside the disk, then the perpendicular bisector of PQ is not an axis of symmetry, where Q is any point on the circumference. We conclude that the n points must be the vertices of a regular n-gon. It is easy to check that the conditions of the problem are satisfied when S consists of one or two points or if S contains $n \geq 3$ points which are the vertices of a regular n-gon.

Double deck

Gerry Myerson: Assume to the contrary that no person ever holds two cards labelled with the same number. Then the two people with 25 will hold them for eternity and a card labelled 24 can only be passed by one of the people holding a 25. So it can be passed at most twice, after which it also stays put for eternity. Now a card labelled 23 can only be passed by one of the people holding a 24 or a 25. Once the cards labelled 24 have settled down, each 23 can only be passed at most four times before it also becomes stationary. And so it goes for the cards labelled 22, 21, and so on, down to 14. After a finite amount of time, the twenty-four cards labelled from 14 up to 25 must be in the hands of 24 different players, where they will remain forever. From this point in time, the 25th player can never hold a card labelled with a number larger than 13, and hence, can never pass a 13. On the other hand, the other players must always pass a 13, so both cards labelled 13 must end up in the hands of the 25th player. This contradicts the assumption that no person ever holds two cards labelled with the same number.

Monks on a mountain

Ian Wanless:

(1) We need the assumption that dawn and dusk are at the same time on the two different days and that the monk's altitude is a continuous function of time. Measure time in such a way that dawn corresponds to $t = 0$ and dusk corresponds to $t = 1$. Then consider the function $F(t)$ to be the monk's altitude at time t on Monday minus the monk's altitude at time t on Tuesday. If the summit is a height h above the base of the mountain, then we have $F(0) = -h$ and $F(1) = h$. Then the intermediate value theorem guarantees that there exists $0 < T < 1$ at which $F(T) = 0$. This implies that the monk's altitude at time T on Monday is the same as the monk's altitude at time T on Tuesday.

(2) We can assume that there are no segments of the path which are flat. Define an allowable position to be one where both monks are at the same altitude with at least one of them at a peak, trough or sea level. We now form a state space, where each state consists of an allowable position as well as the current directions of travel for both monks. Note that this state space must be finite and that the initial state has the monks at opposite ends of the mountain chain and heading towards each other. The monks then walk in time steps

that correspond to transitions in the state space, according to the following rules.

(a) The monks walk in their current direction until exactly one of them reaches a peak or trough, or until both are at sea level. This will be their position for the next state.

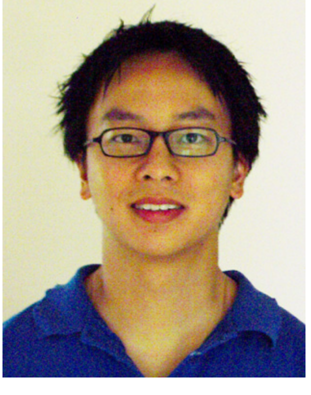
(b) If both monks reach sea level, they maintain their directions. Otherwise, the monk who reaches the peak or trough will maintain his direction, while the other monk turns around. This determines their directions in the next state and ensures that the monks are either both ascending or both descending at all times.

(c) These rules are followed until the monks meet each other or reach sea level.

If the monks meet, then they offer a prayer of thanks and then undertake each other's course up to that point in reverse. This results in them successfully traversing the mountain chain. If the monks return to sea level, but not both to their original positions, then they are separated by a strictly shorter mountain chain and we can solve the problem by induction on the number of peaks.

Suppose that there exist times $s < t$ where the monks are in the same positions but both of them are travelling in opposite directions. Furthermore, suppose that t is minimal with respect to this condition. Note that $t > s+2$, since at most one monk can reverse direction at each time step, but both have reversed between s and t. The crucial observation is that (a) and (b) not only determine the next state, but also the previous one. This time reversibility leads to a contradiction, since the monks were in the same positions at times $s+1 < t-1$ but travelling in opposite directions. This contradicts the minimality of t, and it follows that the monks can never simultaneously return to their original positions.

Another corollary of time reversibility is that a particular state can never be revisited. Given that the state space is finite, this means that condition (c) must eventually be satisfied. Therefore, the path can always be traversed by the monks.



Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.