How to multiply and divide triangles

Maurice Craig*

Introduction

'Rational' trigonometry [4] invokes a novel vocabulary. New words promote concepts that, although easily expressible in Euclidean terms, arguably simplify the subject. Thus Pythagoras' Theorem, expressed in 'quadrances' (squared lengths), reduces to a linear equation.

Pedagogical debate need not inhibit other inquiry. Here I relate triangles, whose sides have integral quadrances, to positive binary quadratic forms with integer coefficients. Geometry makes no special study of triangles with commensurate sides, nor do some of the simplest triangles (e.g. the right-angled isosceles) fit that type. But there is an extensive theory of integral quadratic forms and hence, implicitly, of triangles with integer quadrances.

Section 2 recalls basic notions about quadratic forms, while Section 3 sets up their correspondence with triangles. The fourth section then examines the trigonometric counterpart of composition identities.

Quadratic forms

A real binary quadratic form is an expression $f(x, y) = ax^2 + bxy + cy^2$. Completing the square shows that $f(x, y)$ is positive definite if, and only if, the leading coefficient $a > 0$ and the discriminant $b^2 - 4ac = -d < 0$.

Number theory considers forms whose coefficients a, b, c and variables x, y are integers. Suppose $g(x, y) = f(\alpha x + \beta y, \gamma x + \delta y)$, where $\alpha, \beta, \gamma, \delta$ are integers with $\alpha\delta - \beta\gamma = \pm 1$. We say f and g are equivalent, properly or improperly according as the sign is plus or minus. Properly equivalent forms make up a proper equivalence class (or simply a class) of forms.

The contraction (a, b, c) is often used. Thus, we can remark that $(1, 0, 2)$ and $(2, 0, 1)$ are mutually equivalent, both properly and improperly, although verification might mean restoring the suppressed variables. Every form considered below is positive, and hence [3, Section 92] properly equivalent to a reduced form: one with $|b| \le a \le c$. This reduced form is unique if, when $|b| = a$ or $a = c$, we require further that $b \ge 0$. (Caution: [2], [3] treat forms $(a, 2b, c)$.)

Several different (and hence properly inequivalent) reduced forms can have the same d. For instance, the reduced forms with $d = 23$ are $(1, 1, 6)$, $(2, 1, 3)$ and $(2, -1, 3)$. Accordingly, three classes comprise forms with discriminant (-23).

Received 30 March 2007; accepted for publication 8 August 2007
*E-mail: towenaar@optusnet.com.au

Classes dominate the theory of Diophantine equations $f(x, y) = M$. Equivalent forms take the same values for integer x and y, so the equations for a whole class can be treated together. (See also [3, Section 86(ii)].) Moreover, solutions for composite M are built up from solutions of equations $g(x, y) = N$. Here, N denotes a divisor of M, while g represents a form with the same discriminant as f, though not necessarily in the same class.

This reduction, from M to N, depends on composition of forms. For a comprehensive treatment, see [1], [2], and especially [3, Section 111]. The basic rule for performing composition reads

$$(a, b, Ac)(A, b, ac) = (aA, b, c).$$

Thus the compound of two forms, respectively representing a and A, will itself represent the product aA. All three forms in this shorthand equation have the same central coefficient b, and the same discriminant $b^2 - 4aAc = -D$. Rewritten with the variables restored, the result is $F(x, y)G(s, t) = H(X, Y)$, where $X = xs - cyt$, $Y = axt + Ays + byt$ and $F(x, y) = ax^2 + bxy + Acy^2$ etc.

Example 1. The composite of $(2, 1, 3)$ with $(2, -1, 3)$ is $(1, 1, 6)$. For, $(2, -1, 3)$ is properly equivalent to $(3, 1, 2)$, so the formula produces $(6, 1, 1)$, in the same class as $(1, 1, 6)$. More generally, $(a, b, c)(c, b, a)$ is a principal form, representing unity. To 'divide' by (a, b, c) is to multiply by (c, b, a), in its 'reciprocal' class.

Forms and triangles

Let U, V, W and u, v, w be the angles and respective opposite sides of a triangle. The connection with forms flows from the cosine rule $w^2 = u^2 + v^2 - 2uv \cos W$, reformulated in [4] as the 'cross law'. Thus, after rearrangement, squaring and use of the Pythagorean identity relating sine and cosine, we arrive at

$$(u^2 + v^2 - w^2)^2 - 4u^2v^2 = -(2uv \sin W)^2.$$

Consequently, $(u^2, u^2 + v^2 - w^2, v^2)$ is a positive definite (real) binary quadratic form. Our main result reformulates this finding more exactly.

Proposition 1. *The (row–vector) mapping* $[a, b, c] \mapsto [p, q, r] = [a, c, a - b + c]$ *defines a 3-to-1 correspondence between positive definite integral forms* (a, b, c) *and triangles whose sides, taken anti-clockwise, have integer quadrances* p, q, r.

Proof. We have $r = f(1, -1) > 0$. Moreover (with $b^2 - 4ac = -d$),

$$d = (\sqrt{p} + \sqrt{q} + \sqrt{r}\,)(-\sqrt{p} + \sqrt{q} + \sqrt{r}\,)(\sqrt{p} - \sqrt{q} + \sqrt{r}\,)(\sqrt{p} + \sqrt{q} - \sqrt{r}\,) > 0.$$

The first factor is positive, as are at least two others, e.g. the last two, if $p = \max(p, q, r)$. So all factors are positive, ensuring that every putative side is less than the sum of the other two.

Triangles with sides p, q, r and the same orientation are necessarily congruent. Choosing anti-clockwise orientation as standard lets us associate, with each form (a, b, c), a triangle — having integer quadrances — unique up to congruence.

Conversely, given a triangle T whose integer quadrances, in anti-clockwise order, are p, q, r, by taking $[a, b, c] = [p, p + q - r, q]$ we obtain a positive form (a, b, c)

that maps to T, whence the mapping is surjective. Three such forms arise, depending on whether $a = p$, q or r. (In exceptional cases the forms coalesce and the correspondence is no longer 3-to-1.)

Example 2. The first part of the proof follows also from the stock trigonometric formulae $u = v\cos W + w\cos V$ etc.

Example 3. Let the quadrances for T satisfy $p \le q \le r$. If no interior angle is obtuse then $(p, p+q-r, q)$ is a reduced form. For $p+q-r \ge 0$ (Pythagoras!). However, the triangle with sides $3, 4, 6$ gives $(9, -11, 16)$, not a reduced form.

Composition of triangles

A further development now intervenes, annulling the ambiguity in Proposition 1. For, the three forms $(p, p+q-r, q)$, $(q, -p+q+r, r)$, $(r, r+p-q, p)$ are easily shown properly equivalent. Hence the mapping defined above, while 3-to-1 as a correspondence between forms and triangles, is potentially a 1-to-1 mapping of classes. To realise this potential, we define triangle classes accordingly: call triangles equivalent if their associated forms are so. Corresponding to the equal discriminants of equivalent forms, we have equal areas (or 'quadreas', in rational trigonometry) for equivalent triangles.

Composition of forms implies a law for compounding triangles. Thus, from the triangles with quadrances $\{2, 3, 4\}$ and $\{2, 3, 6\}$, Example 1 produces a further triangle, their compound, with the same quadrea but with quadrances $\{1, 6, 6\}$. This illustration typifies composition of two triangles with integer quadrances and the same quadrea.

Example 4. Let R be the interior angle opposite r for (a, b, Ac). Define R', R'' similarly for (A, b, ac) and (aA, b, c). By the cosine rule we find $R = R' = R''$.

Instances like $(1, 0, 1)$ and $(1, 1, 1)$ will alert experts that integer-quadrance triangles must relate somehow to the point-lattice representation of quadratic forms [2, Section 121–124]; [3, Section 120]. The recondite properties of rational trigonometry's simplest figures show, nevertheless, how thin is the divide between elementary and advanced mathematics. Trigonometry was not meant to be easy!

References

[1] Dickson, L.E. *History of the Theory of Numbers, Vol. III: Quadratic and Higher Forms.* Chelsea, NY.

[2] Mathews, G.B. *Theory of Numbers.* Chelsea, NY.

[3] Smith, H.J.S. *Report on the Theory of Numbers.* Chelsea, NY.

[4] Wildberger, N.J. (2006). Trisecting the equilateral triangle with rational trigonometry. *Gaz. Aust. Math. Soc.* **33,** 333–337. See also http://www.austms.org.au/Gazette/ (accessed 23 August 2007) and http://wildegg.com/authors.htm (accessed 27 August 2007).