From Heron's formula to a characteristic property of medians in the triangle

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Abstract

In an arbitrary triangle, the medians form a triangle as well. We investigate whether this simple property holds for other intersecting cevians as well, and show that the answer is no.

It is a well-established fact in Euclidean geometry that, given a triangle ABC with sides a, b and c, its medians m_a, m_b and m_c also form a triangle called the median triangle. In fact, it can be shown that the median triangle exists in hyperbolic geometry as well, see [3]. Returning to the traditional setting, the existence of the median triangle can be proved in many ways, in particular, through a Heron-type formula which relates the area of ABC only in terms of its medians. We have:

$$\text{Area}(ABC) = \tfrac{4}{3}\sqrt{s(s-m_a)(s-m_b)(s-m_c)},$$

where $s = (m_a + m_b + m_c)/2$. Assuming without loss of generality that $a \geq b \geq c$, we get $m_a \leq m_b \leq m_c$. Furthermore, the existence of the square root gives $s > m_c$, which proves that the triangle inequalities are indeed satisfied for the medians. If the expression inside the square root comes out negative, we go beyond the familiar Euclidean plane and end up in a Lorentz or Minkowski plane. A nice discussion of what happens in this case can be found in [2].

The reader might very well ask: is there a bisector triangle or an altitude triangle (in the Euclidean plane) as well? The answer is no, and one can easily construct examples of triangles for which neither the bisectors nor the altitudes satisfy the triangle inequality; see for example [1]. It is as if the medians are 'the chosen ones' to form a new triangle among all intersecting cevians in the given triangle! Could this be true? Our investigations lead us to the surprising answer of yes! We can indeed prove a certain uniqueness property of the median triangle. Moreover, and to add to the surprise, we will use a bit of algebra and analysis in our proof. Because of this, the proof if elementary will seem technical at times. Of course, it would be interesting to know whether any geometric and more intuitive proofs exists as well, but so far we have been unable to find any.

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We start by setting up some notation. Let $0 < p,\ q,\ r < 1$ be three arbitrary numbers. Depending on the context we will use $|\cdot|$ to denote either the length of a segment or the absolute value of a real number. Given the triangle ABC with sides a, b, c, we consider the points $A' \in (BC)$, $B' \in (CA)$, and $C' \in (AB)$ such that AA', BB', CC' are intersecting cevians, and $|BA'| = pa$, $|CB'| = qb$, and $|AC'| = rc$. Ceva's theorem then restricts one of the parameters p, q, r to be dependent on the other two, since

$$\frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1 \iff \frac{pqr}{(1-p)(1-q)(1-r)} = 1,$$

or

$$r = \frac{(1-p)(1-q)}{pq + (1-p)(1-q)}. \tag{1}$$

We are now ready to state our main result.

Theorem 1 (Uniqueness of medians)**.** *If for all triangles ABC in the plane, three given intersecting cevians AA', BB', CC' form a triangle, then these cevians must be medians.*

Let us first clarify the statement of our theorem. We allow the sides a, b and c of the triangle to be arbitrary numbers strictly larger than zero, restricted only by the triangle inequality

$$b + c > a > |b - c|. \tag{2}$$

The parameters $0 < p, q < 1$, and in particular r given by (1), are fixed, that is we require the cevians to cut the three sides of any triangle in the same ratio.

Using the cosine formula in triangles ABA' and ABC, we can write

$$|AA'|^2 = c^2 + (pa)^2 - 2c(pa)\cos(\angle B) = c^2 + p^2a^2 - p(a^2 + c^2 - b^2),$$

or

$$|AA'| = \sqrt{p^2a^2 - p(a^2 + c^2 - b^2) + c^2}.$$

Similarly, we have

$$|BB'| = \sqrt{q^2b^2 - q(b^2 + a^2 - c^2) + a^2},$$

$$|CC'| = \sqrt{r^2c^2 - r(c^2 + b^2 - a^2) + b^2}.$$

We can now reformulate Theorem 1 and concentrate on proving the restated result.

Theorem 1*. *Let $0 < p, q < 1$ be fixed and $r = r(p, q)$ be defined by (1). If $|AA'| + |BB'| > |CC'| > ||AA'| - |BB'||$ for all $a, b, c > 0$ satisfying (2), then $p = q = r = 1/2$.*

Proof. We simplify further our statement. Let $u = b/a, v = c/a$ and define

$$S = \{(u, v) \in (0, \infty) \times (0, \infty) \colon u + v > 1 > |u - v|\}.$$

For fixed $0 < p, q < 1$, and $(u, v) \in S$, define

$$\begin{aligned}\phi(p,q,u,v) &= (pq + (1-p)(1-q)) \\ &\quad \times \left(\sqrt{p^2 - p(1+v^2-u^2)+v^2} + \sqrt{q^2u^2 - q(1+u^2-v^2)+1}\right), \\ \psi(p,q,u,v) &= \sqrt{\begin{aligned}&(1-p)^2(1-q)^2v^2 - (pq+(1-p)(1-q)) \\ &\quad \times (1-p)(1-q)(u^2+v^2-1) + u^2(pq+(1-p)(1-q))^2\end{aligned}}, \\ \delta(p,q,u,v) &= (pq + (1-p)(1-q)) \\ &\quad \times \left|\sqrt{p^2 - p(1+v^2-u^2)+v^2} - \sqrt{q^2u^2 - q(1+u^2-v^2)+1}\right|.\end{aligned}$$

Note that if we factor out a^2 in the expressions under the square roots that define $|AA'|$, $|BB'|$, $|CC'|$, substitute the expression (1) defining r in the length formula of $|CC'|$, and multiply through with $pq + (1-p)(1-q)$ in the triangle inequality satisfied by the intersecting cevians, we arrive at the following.

Claim. If $\phi(p,q,u,v) > \psi(p,q,u,v) > \delta(p,q,u,v)$ *for all* $(u,v) \in S$ *then* $p = q = \frac{1}{2}$.

To prove this claim we use an invariance property of the set S and some symmetry properties of the functions ϕ, ψ and δ.

First, note that S is an open strip in the first quadrant of the uv-plane that is bordered by (and does not include) the slanted lines $v = u + 1, v = u - 1$, and $v = 1 - u$. The first two of the border lines intersect the third one at the points $P = (0, 1)$ and $Q = (1, 0)$, which belong to the boundary ∂S of our strip (see Figure 1).

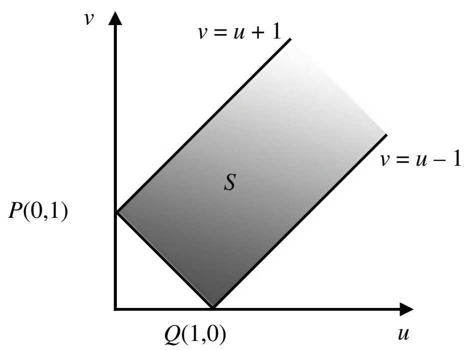

Figure 1.

It is easy to show that S satisfies the following invariance property

$$(u, v) \in S \;\leftrightarrow\; \left(\frac{1}{u}, \frac{v}{u}\right) \in S. \tag{3}$$

Second, a series of elementary, yet tedious, computations give the symmetry properties

$$\begin{aligned}\phi\left(1-q,1-p,\frac{1}{u},\frac{v}{u}\right)&=\frac{1}{u}\phi(p,q,u,v),\\ \psi\left(1-q,1-p,\frac{1}{u},\frac{v}{u}\right)&=\frac{1}{u}\psi(p,q,u,v),\\ \delta\left(1-q,1-p,\frac{1}{u},\frac{v}{u}\right)&=\frac{1}{u}\delta(p,q,u,v).\end{aligned} \tag{4}$$

The upshot of the invariance and symmetry properties (3) and (4) is that for any equation satisfied by the pair (p,q) we get another 'dual equation' by replacing (p,q) with $(1-q,1-p)$. Clearly, ϕ, ψ, and δ are continuous functions (of the variables u,v) on S. If we pass to the limit, as $(u,v)\to P\in\partial S$, the second inequality in the condition of the Claim becomes

$$\begin{aligned}(1-p)(1-q)&=\psi(p,q,0,1)\geq\delta(p,q,0,1)\\ &=(pq+(1-p)(1-q))\left|\sqrt{p^2-2p+1}-1\right|,\end{aligned}$$

or, after simplifying,

$$(1-p)^2(1-q)\geq p^2q. \tag{5}$$

The dual inequality of (5) is

$$q^2p\geq(1-q)^2(1-p). \tag{6}$$

Similarly, when $(u,v)\to Q\in\partial S$, the first inequality in the condition of the Claim gives

$$(pq+(1-p)(1-q))\left(p+\sqrt{q^2-2q+1}\right)\geq pq+(1-p)(1-q),$$

or

$$p\geq q. \tag{7}$$

Note that the dual of (7) gives the same inequality.

We are now very close to the conclusion. Let $s=p/(1-p)$ and $t=q/(1-q)$. From (5), (6), (7) we have

$$s^2t\leq 1,\quad st^2\geq 1,\quad\text{and}\quad s\geq t,$$

or

$$\frac{1}{st}\geq s\geq t\geq\frac{1}{st}.$$

Thus, $s=t=1$ which is equivalent to $p=q=\frac{1}{2}$ (and hence also $r=\frac{1}{2}$). The proof is complete.

References

[1] Bényi, Á. (2003). A Heron-type formula for the triangle. *The Mathematical Gazette* July 2003, 324–326.

[2] Kendig, K. (2000). Is a 2000-year-old formula still keeping some secrets? *American Mathematical Monthly* May 2000, 402–415.

[3] Leonard, I.E., Lewis, J.E., Liu, A. and Tokarsky, G. (2004). The median triangle in hyperbolic geometry. *Mathematics Magazine* October 2004, 293–297.