Communications

Rolle to Cauchy

J.J. Koliha* **and Peng Zhang****

The mean value theorem is one of the cornerstones of calculus, yet for its proof most calculus books suddenly pull a complicated looking function

$$F(x) = f(x) - f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

out of the hat in the quest of reducing the problem to Rolle's theorem on $[a, b]$.

Not surprisingly many first year students of calculus feel intimidated by such an *ad hoc* approach, and miss the simple connection between the intuitively more obvious Rolle's theorem, and its more complicated cousins, Lagrange's and Cauchy's mean value theorems.

In this note we offer a delightfully simple do-it-yourself approach to Cauchy's mean value theorem which provides the missing motivation and goes straight from 'Rolle to Cauchy'. We need to assume the following version of Rolle's theorem (see for instance [1, p.255]).

Theorem 1 (Rolle's theorem)**.** *Let a function $f\colon [a, b] \to \mathbb{R}$ be continuous in $[a, b]$, differentiable in (a, b), and let $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

Now assume that functions $f, g\colon [a, b] \to \mathbb{R}$ are continuous in $[a, b]$ and differentiable in (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. In order to apply Rolle's theorem, we set

$$G(x) = f(x) - Ag(x)$$

with $A \in \mathbb{R}$ chosen to satisfy $G(a) = G(b)$. First we show that $g(a) \neq g(b)$. If we had $g(a) = g(b)$, then by Rolle's theorem there would exist $d \in (a, b)$ such that $g'(d) = 0$. But this contradicts our assumption about the derivative of g. Hence $g(a) \neq g(b)$. Solving $G(a) = G(b)$ gives $A = (f(b) - f(a))/(g(b) - g(a))$. The function G satisfies the assumptions of Rolle's theorem, and hence there exists $c \in (a, b)$ with $G'(c) = 0$, that is,

$$G'(c) = f'(c) - Ag'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g'(c) = 0.$$

Thus we have proved

Theorem 2 (Cauchy's mean value theorem)**.** *Let functions $f, g\colon [a, b] \to \mathbb{R}$ be continuous in $[a, b]$, differentiable in (a, b), and let $g'(x) \neq 0$ on (a, b). Then there*

*Department of Mathematics and Statistics, The University of Melbourne, VIC 3010.
E-mail: j.koliha@ms.unimelb.edu.au
**E-mail: p.zhang@ugrad.unimelb.edu.au

exists a point $c \in (a,b)$ *such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Setting $g(x) = x$ in the preceding theorem, we obtain

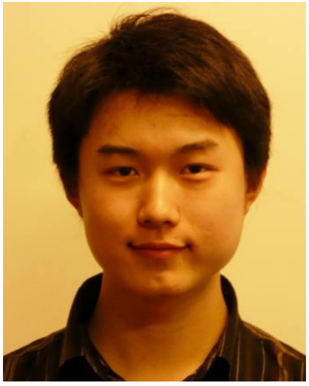
Theorem 3 (Lagrange's mean value theorem)**.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous in* $[a,b]$ *and differentiable in* (a,b)*. Then there exists a point* $c \in (a,b)$ *such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

References

[1] Thomas, G.B., Weir, M.D., Hass, J.D. and Giordano, F.R. (2005). *Thomas' Calculus*, 11th edn. Pearson/Addison Wesley, Boston.

J.J. Koliha



Peng Zhang

J.J. Koliha is Associate Professor in the Mathematics and Statistics Department of the University of Melbourne. He specialises in Functional Analysis. He says that every year he learns from his students as much as they learn from him.

Peng Zhang is an international student currently in the first year of the Commerce Actuarial Studies at the University of Melbourne. He has been interested in mathematics ever since he could write his own name. He is of Chinese background, born in Beijing.