Technical papers

On the rate of convergence of Wallis' sequence

Eugen Păltănea

Abstract

Recent papers published in the *Gazette* deal with the asymptotic behaviour of Wallis' sequence $W_n = \prod_{k=1}^{n} 4k^2/(4k^2-1)$. Our purpose is to interpret the well-known formula of the rate of convergence: $W_n = \pi/2 - \pi/8n + o(1/n)$ as $n \to \infty$, in the language of the sequences of definite integrals.

Key words: the rate of convergence, sequences of definite integrals.
MSC: 26A15, 26A42, 40A05, 40A20.

Introduction

The famous Wallis' sequence $(W_n)_{n\geq 1}$ is defined by:

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} = \frac{\pi}{2}\,\frac{\int_0^{\pi/2} \sin^{2n+1} x\,dx}{\int_0^{\pi/2} \sin^{2n} x\,dx}, \qquad n \geq 1.$$

As shown by Hirshhorn [1], and earlier by Vernescu [8],

$$W_n = \frac{\pi}{2} - \frac{\pi}{8n} + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

In this paper, using the integral expression of W_n, we show that the limit

$$\lim_{n\to\infty} n\left(\frac{\pi}{2} - W_n\right) = \frac{\pi}{8} \tag{1}$$

follows from general properties of some sequences of definite integrals.

Basic results on the convergence of some sequences of definite integrals

We shall investigate the asymptotic behavior of the sequence of integrals $I_n = \int_a^b f^n(x)\,dx$, $n \in \mathbb{N}$, where $f\colon [a,b] \to \mathbb{R}$ is an integrable function. The following elementary theorem (see [5] for the proof) refers to the convergence of the sequence $(I_{n+1}/I_n)_{n\geq 1}$.

Theorem 1. *Let $f\colon [a,b] \to \mathbb{R}_+$ be a positive continuous function with $\|f\| = \max_{x\in[a,b]} f(x)$. Let us denote $I_n = \int_a^b f^n(x)\,dx$, $n \in \mathbb{N}$. Then $(I_{n+1}/I_n)_{n\geq 1}$ is an increasing sequence with:*

$$\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = \|f\|.$$

The fact that the sequence $(I_{n+1}/I_n)_{n\geq 1}$ is monotonic increasing is a consequence of Bunyakovsky's inequality.

Now let us discuss the special case when f reaches its maximum $\|f\|$ in a unique point. We begin with the following useful statement.

Lemma 1. *Let $f\colon [a,b] \to \mathbb{R}_+$ be a positive continuous function with the property that there is a unique point $c \in [a,b]$ such that $\|f\| = f(c)$. Also let $g\colon [a,b] \to \mathbb{R}$ be a continuous function. Then the sequence:*

$$x_n = \frac{\int_a^b f^{n+1}(x)g(x)\,dx}{\int_a^b f^n(x)\,dx}, \qquad n \geq 1$$

converges to $\|f\|g(c)$.

Proof. Let us choose an arbitrary $\varepsilon > 0$. Since f and g are continuous at c it follows that there is $[u,v] \subset [a,b]$, with $u < v$ and $c \in [u,v]$, such that

$$|f(x)g(x) - f(c)g(c)| < \frac{\varepsilon}{2}, \quad \text{for all } x \in [u,v].$$

Let us denote $m := \max\{f(x) \mid x \in [a,b]\backslash[u,v]\}$. By the assumed uniqueness of the maximum point c we have $m < \|f\| = f(c)$. From the continuity of f at c, for a fixed $m_1 \in (m, \|f\|)$ there exists an interval $[s,t] \subset [a,b]$, with $s < t$, such that $f(x) \geq m_1$, for all $x \in [s,t]$. Also, since $(m/m_1)^n \to 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $2A(b-a)/(t-s)(m/m_1)^n < \varepsilon/2$, for all $n \geq n_\varepsilon$, where $A := \max_{x\in[a,b]} |f(x)g(x)|$. Hence, for any $n \geq n_\varepsilon$, we have:

$$\begin{aligned}
|x_n - f(c)g(c)| &\leq \frac{\int_a^b f^n(x)|f(x)g(x) - f(c)g(c)|\,dx}{\int_a^b f^n(x)\,dx} \\
&= \frac{\int_{[a,b]\backslash[u,v]} f^n(x)|f(x)g(x) - f(c)g(c)|\,dx}{\int_a^b f^n(x)\,dx} \\
&\quad + \frac{\int_u^v f^n(x)|f(x)g(x) - f(c)g(c)|\,dx}{\int_a^b f^n(x)\,dx} \\
&\leq \frac{2A\int_{[a,b]\backslash[u,v]} f^n(x)\,dx}{\int_s^t f^n(x)\,dx} + \frac{\varepsilon}{2}\frac{\int_u^v f^n(x)\,dx}{\int_a^b f^n(x)\,dx} \\
&\leq 2A\frac{b-a}{t-s}\left(\frac{m}{m_1}\right)^n + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

This shows that $\lim_{n\to\infty} x_n = f(c)g(c) = \|f\|g(c)$.

Below we present a deduction of the rate of convergence of the sequence $(I_{n+1}/I_n)_{n\geq 1}$ for twice-differentiable functions with continuous second derivatives.

Theorem 2. *Let $f\colon [a,b] \to \mathbb{R}_+$ be a positive twice-differentiable function with continuous second derivative. Assume that $f'(x) > 0$, for all $x \in [a,b)$ and $f''(b) \neq 0$. Then the sequence*

$$y_n = n\left(f(b) - \frac{\int_a^b f^{n+1}(x)\,dx}{\int_a^b f^n(x)\,dx}\right), \qquad n \geq 1$$

is convergent and we have:

$$\lim_{n\to\infty} y_n = \begin{cases} f(b), & \text{if } f'(b) \neq 0, \\ \dfrac{f(b)}{2}, & \text{if } f'(b) = 0. \end{cases}$$

Proof. It is obvious that b is the unique maximum point of the function f. We have

$$\lim_{x\to b^-} \frac{f(x)-f(b)}{f'(x)} = 0$$

(since $f''(b) \neq 0$ we use l'Hôpital's rule when $f'(b) = 0$). Thus, the function $g\colon [a,b] \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} \dfrac{f(x)-f(b)}{f'(x)}, & x \in [a,b), \\ 0, & x = b, \end{cases}$$

is continuous. Also, we obtain

$$\lim_{x\to b^-} \frac{f(x)-f(b)}{(f'(x))^2} = \begin{cases} 0, & f'(b) \neq 0, \\ \dfrac{1}{2f''(b)}, & f'(b) = 0. \end{cases}$$

It follows that g is differentiable with continuous derivative on $[a,b]$ and we have

$$g'(b) = \lim_{x\to b^-} g'(x) = \begin{cases} 1, & f'(b) \neq 0, \\ \frac{1}{2}, & f'(b) = 0. \end{cases}$$

Therefore, using the method of integration by parts, we can write

$$\begin{aligned}(n+1)\int_a^b f^n(x)\big(f(b)-f(x)\big)\,dx &= -(n+1)\int_a^b f^n(x)f'(x)g(x)\,dx \\ &= -f^{n+1}(x)g(x)\Big|_a^b + \int_a^b f^{n+1}(x)g'(x)\,dx.\end{aligned}$$

Thus, we obtain

$$y_n = \frac{n}{n+1}\left(\frac{f^{n+1}(a)g(a)}{\int_a^b f^n(x)\,dx} + \frac{\int_a^b f^{n+1}(x)g'(x)\,dx}{\int_a^b f^n(x)\,dx}\right).$$

Let us choose $c \in (a,b)$. Since f is increasing on $[a,b]$, $f(x) \geq f(c)$, for all $x \in [c,b]$ and $f(a)/f(c) \in [0,1)$. From the obvious inequalities

$$0 \leq \frac{f^n(a)}{\int_a^b f^n(x)\,dx} < \frac{f^n(a)}{\int_c^b f^n(x)\,dx} < \frac{1}{b-c}\left(\frac{f(a)}{f(c)}\right)^n$$

we get $\lim_{n\to\infty} f^n(a)/(\int_a^b f^n(x)\,dx)$. Further, using Lemma 1, we find

$$\lim_{n\to\infty} \frac{\int_a^b f^{n+1}(x)g'(x)\,dx}{\int_a^b f^n(x)\,dx} = f(b)g'(b).$$

Hence, the sequence (y_n) is convergent with:

$$\lim_{n\to\infty} y_n = f(b)g'(b) = \begin{cases} f(b), & f'(b) \neq 0, \\ \dfrac{f(b)}{2}, & f'(b) = 0. \end{cases}$$

We have thus proved the theorem.

Computing the rate of convergence of Wallis' sequence

Let us consider the function $f\colon [0,\pi/2] \to [0,1]$, $f(x) = \sin x$ and the sequence of Riemann integrals

$$I_n = \int_0^{\pi/2} f^n(x)\,dx, \quad \text{for } n \geq 1.$$

We shall begin with a method (see [5]) which is based on the well-known recurrence relation:

$$I_{n+2} = \frac{n+1}{n+2}\, I_n. \tag{2}$$

By Theorem 1, we have for any positive integer n the following inequality:

$$\frac{I_{2n}}{I_{2n-1}} \leq \frac{I_{2n+1}}{I_{2n}} \leq \frac{I_{2n+2}}{I_{2n+1}}.$$

Hence, from (2) we obtain:

$$\frac{2n}{2n+1} = \frac{I_{2n+1}}{I_{2n-1}} \leq \left(\frac{I_{2n+1}}{I_{2n}}\right)^2 \leq \frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}.$$

Therefore we find:

$$\frac{\pi}{2}\sqrt{\frac{2n}{2n+1}} \leq W_n \leq \frac{\pi}{2}\sqrt{\frac{2n+1}{2n+2}}.$$

Thus, the following inequalities arise:

$$\frac{\pi/4}{\sqrt{1+1/n}\left(\sqrt{1+1/n}+\sqrt{1+1/2n}\,\right)} \leq n\left(\frac{\pi}{2} - W_n\right) \leq \frac{\pi/4}{\sqrt{1+1/2n}\left(1+\sqrt{1+1/2n}\,\right)}, \quad \text{for all } n \in \mathbb{N}.$$

Consequently, limit (1) exists.

But we have not exposed a 'general method' because the particular recurrence relation (2) of (I_n) is used in the above proof. A more instructive general method of obtaining (1) is based entirely on Theorem 2. Thus, since $f'(x) > 0$, for all $x \in [0,\pi/2)$, $f'(\pi/2) = 0$ and $f''(\pi/2) \neq 0$, we have:

$$\lim_{n\to\infty} n\left(1 - \frac{I_{2n+1}}{I_{2n}}\right) = \frac{1}{2}\lim_{n\to\infty}(2n)\left(f\left(\frac{\pi}{2}\right) - \frac{I_{2n+1}}{I_{2n}}\right) = \frac{1}{2}\cdot\frac{f(\pi/2)}{2} = \frac{1}{4}.$$

If we multiply by $\pi/2$, then we obtain the limit (1).

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Transilvania University of Braşov, Braşov, Romania
E-mail: epaltanea@unitbv.ro
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