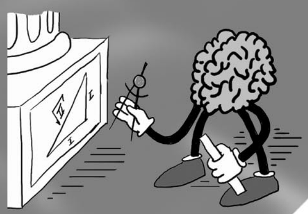

More on π

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Abstract

We give simple proofs of the facts that $\pi < \frac{355}{113}$ and $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

It has been known since Archimedes (c. 250BC) that π is roughly $\frac{22}{7}$, and that $\pi < \frac{22}{7}$. A really neat proof of these facts was found, perhaps by Kurt Mahler in the 1960s, and that is

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi. \qquad (*)$$

It is clear that the integral is positive, and since the denominator of the integrand is at least 1, the integral is less than $\frac{1}{630}$.

It has been known since Zhu Chongzhi (5th century) that an approximation to π, better than $\frac{22}{7}$, is $\frac{355}{113}$, which also is in excess. (π and $\frac{355}{113}$ agree to 6 decimals.)

In a recent article, Stephen Lucas [1] sought a simple integral which is obviously positive, and whose value is $\frac{355}{113} - \pi$. Perhaps the nicest he came up with is

$$\int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{3164(1+x^2)}\,dx = \frac{355}{113} - \pi.$$

I noticed that the idea involved in $(*)$ can be extended to prove not only the desired result, but more.

Consider

$$\int_0^1 \frac{x^{4n}(1-x)^4}{1+x^2}\,dx.$$

To evaluate this integral, we use partial fractions:

$$\frac{x^{4n}(1-x)^4}{1+x^2} = x^{4n+2} - 4x^{4n+1} + 5x^{4n} - 4x^{4n-2} + 4x^{4n-4} - + \cdots + 4 - \frac{4}{1+x^2}.$$

It follows that

$$\begin{aligned}\int_0^1 \frac{x^{4n}(1-x)^4}{1+x^2}\,dx &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots - \frac{1}{4n-1}\right) + \frac{5}{4n+1} \\ &\qquad - \frac{4}{4n+2} + \frac{1}{4n+3} - \pi \\ &= \text{rational } - \pi,\end{aligned}$$

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where the denominator of the rational is (before any cancellation) divisible by all primes up to $4n+3$.

Thus, for example,

$$\int_0^1 \frac{x^{112}(1-x)^4}{1+x^2}\,dx = \frac{P}{113Q} - \pi,$$

where, using my trusty computer I find

$$\begin{aligned} P &= 46\,922\,045\,053\,712\,930\,642\,150\,903\,262\,788\,670\,879\,977\,081\,355\,826,\\ Q &= 132\,174\,785\,996\,436\,457\,344\,235\,486\,484\,552\,040\,071\,436\,191\,175,\\ P < 355Q &= 46\,922\,049\,028\,734\,942\,357\,203\,597\,702\,015\,974\,225\,359\,847\,867\,125 \end{aligned}$$

and

$$0 < \int_0^1 \frac{x^{112}(1-x)^4}{1+x^2}\,dx < \frac{355}{113} - \pi,$$

which gives

$$\pi < \frac{355}{113}.$$

In a similar vein,

$$\int_0^1 \frac{x^{4n+2}(1-x)^4}{1+x^2}\,dx = \pi - \text{rational},$$

where the denominator is (before any cancellation) divisible by all primes up to $4n+5$. In particular,

$$\int_0^1 \frac{x^{70}(1-x)^4}{1+x^2}\,dx = \pi - \frac{R}{71S},$$

where

$$\begin{aligned} R &= 3\,100\,473\,885\,152\,861\,164\,004\,910\,599\,081,\\ S &= 13\,900\,161\,851\,314\,672\,380\,764\,590\,350,\\ R > 223S &= 3\,099\,736\,092\,843\,171\,940\,910\,503\,648\,050 \end{aligned}$$

and

$$0 < \int_0^1 \frac{x^{70}(1-x)^4}{1+x^2} < \pi - \frac{223}{71},$$

from which

$$\pi > \frac{223}{71} = 3\frac{10}{71},$$

a fact known to Archimedes.

References

[1] Lucas, S.K. (2005). Integral proofs that $355/113 > \pi$, *Gaz. Aust. Math. Soc.* **32,** 263–266.