Puzzle corner 3

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 3 is 1 September 2007. The solutions to Puzzle Corner 3 will appear in Puzzle Corner 5 in the November 2007 issue of the *Gazette*.

Milk and tea

Consider a cup of milk and a cup of tea, each containing precisely the same amount of liquid. Three tablespoons from the cup of milk are poured into the cup of tea, and the liquid is thoroughly mixed. Then three tablespoons of this mixture are poured back into the cup of milk. Which is greater now: the percentage of milk in the tea or the percentage of tea in the milk?

Lockers

School lockers numbered from 1 to 100 stand in a row. When the first student arrives at school, she opens all of the lockers. The second student to arrive then closes every second locker starting with locker number 2. The third student then toggles the state of every third locker starting with locker number 3. This continues until 100 students have passed by the lockers. Which lockers are now open?

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Irrational punch

Consider a hole punch that can be centred at any point of the plane and removes those points whose distance from the centre is irrational. How many punches are required to remove every point?

Integral averages

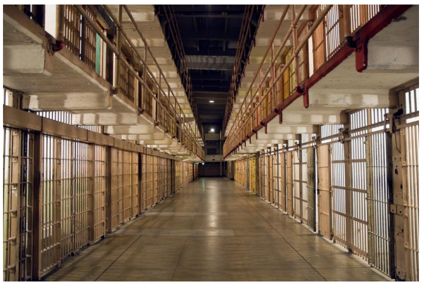
Let $n \geq 2$ be an integer and let T_n be the number of non-empty subsets of $\{1, 2, 3, \ldots, n\}$ with the property that the average of its elements is an integer. Prove that $T_n - n$ is an even number for every value of n.

Silver matrices

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a *silver matrix* if, for each k, the kth row and the kth column together contain all elements of S.

(a) Show that there is no silver matrix for $n = 2007$.
(b) Show that silver matrices exist for infinitely many values of n.

Puzzles for prisoners



(1) One hundred prisoners are on death row. Tomorrow, they will all be lined up in single file, one behind another. Each will have a number from 1 to 100, not necessarily distinct, written on their back. The prisoners can see the numbers on the backs of everyone standing in front of them. They will then be asked in turn, starting with the person at the back of the line, all the way to the person at the front of the line, what they think the number on their back is. If the prisoner answers correctly, then they are allowed to live, otherwise they will be put to death. Every prisoner is able to hear the answer of every other prisoner. Tonight the prisoners are allowed to meet and discuss their strategy. How many lives can the prisoners be guaranteed to save?

(2) Once again, one hundred prisoners are on death row. Tomorrow they will be asked, one by one, to enter a room which contains 100 boxes in a row. The boxes contain the names of all the prisoners, one name to a box. Once inside the room, each prisoner may look inside 50 boxes, but must leave the room exactly as it was before they entered. No further communication is allowed between the prisoners until they have all exited the room. The prisoners will all be executed unless every single one of them manages to find his or her own name inside one of the 50 boxes that they examined. Tonight the prisoners are allowed to meet and discuss their strategy.

Note that if each prisoner looks inside 50 boxes at random, then their probability of survival is a miniscule $\left(\frac{1}{2}\right)^{100} \approx 7.89 \times 10^{-31}$. However, show that if the prisoners choose the correct strategy, then they can all be freed with a probability that exceeds thirty per cent!

Solutions to Puzzle Corner 1

The \$50 book voucher for the best submission to Puzzle Corner 1 is awarded to S. Krass.

Fun with fuses

Solution by Claire Hotan: To measure 45 seconds, light both ends of fuse A and one end of fuse B at the same time. When fuse A burns out precisely 30 seconds must have elapsed, and at this time light the unlit end of fuse B. When fuse B burns out, a further 15 seconds must have elapsed, thereby measuring a total of $30 + 15 = 45$ seconds.

Since one fire will cause a fuse to burn for 60 seconds, maintaining three fires will cause a fuse to burn for 20 seconds. To perform this task, light the fuse at some point in the middle and at one of the ends. Whenever two fires meet and extinguish, light a fire in the middle of an unburnt segment to maintain three fires. If a fire reaches an endpoint and extinguishes, we cut the fuse in an unburnt region and start a fire at one of the new ends. Of course, for this method to work, one may have to perform infinitely many operations infinitely quickly!

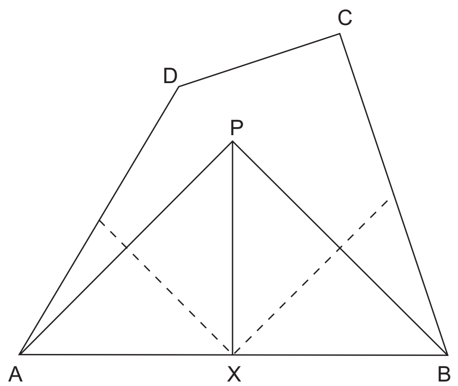
Fuel shortage

Solution by Peter Pleasants: We will solve the problem by induction on n, the number of fuel stations located on the circular route. The base case of one fuel station is entirely trivial. So suppose that the problem is true for n fuel stations and consider the case now that there are $n+1$ fuel stations. Note that there must be at least one fuel station A which contains enough fuel to reach the next fuel station B. Otherwise, it is clear that there would be insufficient fuel in total to make it around the circular route. Using this observation, we know that any car which can make it to A can then refuel and make it to B. Therefore, one may as well consider moving all of the fuel at fuel station B to fuel station A and removing fuel station B entirely. However, this reduces the problem to the case of n fuel stations, which is true by the inductive hypothesis. Therefore, the problem is also true for $n+1$ fuel stations and, by induction, is true for any finite number of fuel stations.

Alternative solution: Consider an imaginary car with a large tank containing more than enough fuel to complete one circuit. Suppose that it begins at some arbitrary point and drives around the circular route, buying up all the fuel at each station that it passes. Note that when the car has completed one circuit, its fuel gauge will read precisely the same amount as when the car began its journey. Therefore, at some point P along the route, the fuel gauge must hit a minimum value. Furthermore, this point P must be at one of the fuel stations. It should be easy to see now that a car starting at point P with an empty tank can make it around the route.

Folding quadrilaterals

Solution by S. Krass: Let us say that a quadrilateral admits a *nice folding* if it can be folded so that the corners all meet at the same point without overlapping. First, we observe that such a quadrilateral must be convex. Now suppose that $ABCD$ admits a nice folding so that the corners meet at a point P. Suppose that folding A to P creates a crease which meets AB at X. Then to avoid overlap, folding B to P must create a crease which meets AB at X also. Since AX folds to PX and BX folds to PX, we have the equalities $AX = PX = BX$. It follows that $\angle APB = 90°$ and by similar reasoning, we also obtain $\angle PBC = \angle CPD = \angle DPA = 90°$. Therefore, the diagonals AC and BD must meet at right angles. Conversely, we observe that any convex quadrilateral in which the diagonals meet at right angles admits a nice folding.



Although this gives a simple geometric condition which determines when a quadrilateral admits a nice folding, we are after a condition which only involves the side lengths of the quadrilateral. This is (almost) possible by the following result.

Lemma 1. *Four points in the plane A, B, C, D satisfy the condition*

$$AB^2 + CD^2 = BC^2 + DA^2$$

if and only if AC is perpendicular to BD.

Proof. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be the position vectors of the points A, B, C, D, respectively. The following chain of equivalent statements yields the desired result.

$$\begin{array}{rrcl} & AB^2 + CD^2 & = & BC^2 + DA^2 \\ \Leftrightarrow & (\mathbf{a}-\mathbf{b})^2 + (\mathbf{c}-\mathbf{d})^2 & = & (\mathbf{b}-\mathbf{c})^2 + (\mathbf{d}-\mathbf{a})^2 \\ \Leftrightarrow & \mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} & = & \mathbf{b}\cdot\mathbf{c} + \mathbf{d}\cdot\mathbf{a} \\ \Leftrightarrow & (\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{d}) & = & 0 \\ \Leftrightarrow & AC & \perp & BD. \end{array}$$

We have already noted that a quadrilateral admits a nice folding if and only if it is convex and its diagonals meet at right angles. Strictly speaking, it is not possible to determine whether or not this is the case given the side lengths alone. In particular, the same set of side lengths can be used to form both convex as well as non-convex quadrilaterals. Even if we restrict ourselves to the case of convex quadrilaterals, it is still not possible to determine whether or not the quadrilateral admits a nice folding from its set of side lengths. For example, if the side lengths are 1, 7, 8 and 4 in that cyclic order, then the quadrilateral admits a nice folding since $1^2 + 8^2 = 7^2 + 4^2$. However, if the side lengths are 1, 4, 7 and 8 in that cyclic order, then such a quadrilateral does not admit a nice folding since $1^2 + 7^2 \neq 4^2 + 8^2$.

Surprisingly enough though, we can determine whether or not a quadrilateral admits a nice folding given only the fact that it is convex as well as its side lengths in cyclic order. The precise statement is as follows:

> A quadrilateral with side lengths a, b, c, d in cyclic order admits a nice folding if and only if it is convex and $a^2 + c^2 = b^2 + d^2$.

Self-referential aptitude test

Solution by Ian Wanless: Depending on your solution to question 20, the following five solutions are consistent.

(A) DADBEDDEDABADBADBEBA
(B) DADBEDDEDABADBADBAEB
(C) DADBEDDEDABADBADBEAC
(D) DADBEDDEDABADCADBEAD
(E) DADBEDDEDABADBADBABE

Of course, Jim Propp, the original proposer of the self-referential aptitude test, expected the correct answer to question 20 to be (E). Jim writes:

> It's no coincidence that this spells out an odd message if spaces are added appropriately. When I created the S.R.A.T. for a party in 1993 or so, it was one of a chain of puzzles, each of which gave clues for the next; in particular, the sentence DAD BEDDED A BAD BAD BABE was one of several hints to the identity of the movie *Fatal Attraction*.

Magic card trick

Solution by S. Krass: For the trick to work with $n \geq 3$ cards, we must have one of the following two arrangements.

- One box contains card 1, one box contains card n and the remaining box contains the cards from 2 to $n - 1$.
- One box contains all cards congruent to 0 modulo 3, one box contains all cards congruent to 1 modulo 3, and the remaining box contains all cards congruent to 2 modulo 3.

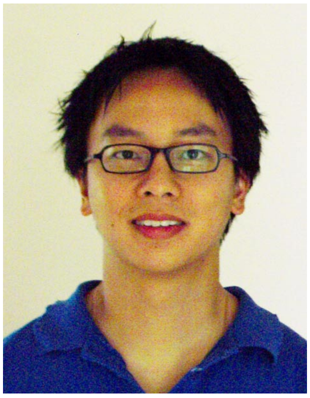
We will prove this by induction on n. Note that the base case when $n = 3$ is trivially true, although the two solutions actually happen to coincide. Now suppose that the result is true for n cards and consider the case when there are $n+1$ cards.

Case 1: The card $n + 1$ is alone in its box.
If 1 is not also alone in its box, then let N be the sum of the largest cards in each of the boxes not containing $n+1$. Note that $n+2 \leq N \leq 2n-1$, so we can achieve the same sum N using the card $n + 1$ and the card $N - n - 1$. Since these two

cards come from a different pair of boxes, we obtain a contradiction. Therefore, the card 1 must also be alone in its box, giving the first solution.

Case 2: The card $n + 1$ is not alone in its box.
Note that if we remove card $n + 1$, then we have a valid solution with n cards. By inspection, this solution must be of the second type. It is now easy to verify that card $n+1$ must then go into the box with all of the cards of the same residue modulo 3.

In conclusion, we note that for 100 cards, the magician has precisely 12 ways to arrange the cards in the boxes, 6 ways for each of the two solutions.



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