On points of contact in infinite chains

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1 Preliminaries

In an earlier investigation we found the locus of the centres of the circles forming an infinite chain and established an analytic method for constructing the chain (see [1]). The assigned task in this paper is to find the locus of the points of contact of the circles in such a chain and find a way to identify these points.

We start with some preliminaries.

Theorem 1 *Two internally tangent circles are given. A third circle is drawn to be tangent to both of them as shown in Figure 1. The locus of all centre points $P(x, y)$ forms an ellipse.*

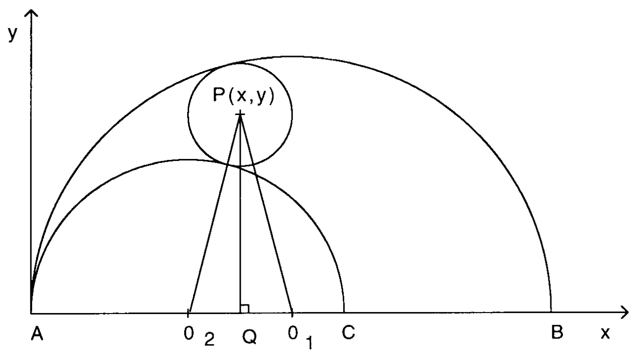

Figure 1.

Proof. Let the centre of the drawn circle be $P(x, y)$ and its radius r.

The centres of the given circles are represented by O_1 and O_2 and their radii by R_1 and R_2 respectively. The point of contact is assigned to be the Origin and the line joining the centres of the given circles the x-axis.

$$\overline{PO_1} + \overline{PO_2} = R_1 - r + R_2 + r = R_1 + R_2 \qquad \text{(Constant)}$$

therefore the locus of P is an ellipse. The Major Axis of the ellipse $2a = R_1 + R_2$ and its foci are O_1 and O_2.

Equation of the ellipse:

$$y^2 = (R_2 + r)^2 - (x - R_2)^2 \tag{1}$$

$$y^2 = (R_1 - r)^2 - (R_1 - x)^2. \tag{2}$$

By eliminating y from (1) and (2) we get:

$$x = \frac{R_1 + R_2}{R_1 - R_2} r. \tag{3}$$

If we substitute x in (3) into either one of (1) and (2) we obtain the equation of the ellipse in the form $y = f(r)$, e.g.,

$$y = \frac{2}{R_1 - R_2}\sqrt{R_1 R_2 (R_1 - R_2 - r) r}. \quad (4)$$

In (3) if r is made the subject; its substitution in (1) or (2) yields

$$y = \frac{2}{R_1 + R_2}\sqrt{R_1 R_2 (R_1 + R_2 - x) x}. \quad (5)$$

□

Theorem 2 *Two externally tangent circles are given. A third circle is drawn to be tangent to both externally. The locus of all centre points $P(x, y)$ forms an hyperbola.*

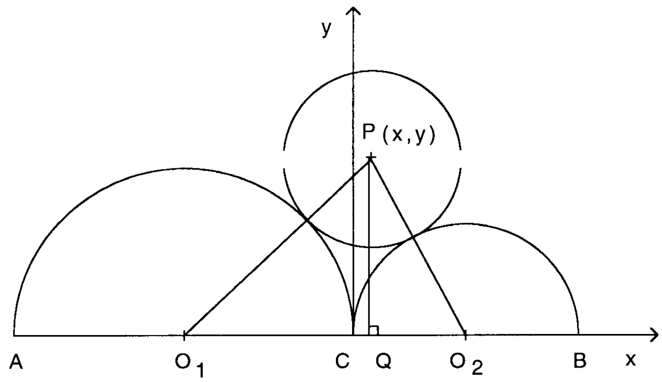

Figure 2.

The proof follows along the same lines as the internal tangency case given in Theorem 1 and the analogous results are given below.

$$x = \frac{R_1 - R_2}{R_1 + R_2} r \quad (6)$$

$$y = \frac{2}{R_1 + R_2}\sqrt{R_1 R_2 (R_1 + R_2 + r) r}, \quad (7)$$

$$y = \frac{2}{R_1 - R_2}\sqrt{R_1 R_2 (R_1 - R_2 + x) x}. \quad (8)$$

Theorem 3 *A line and a circle are tangent to each other. Another circle is drawn tangent to both of them (See Figure 3). The locus of all centre points $P(x, y)$ forms an parabola.*

$$y^2 = (R_2 + r)^2 - (R_2 - r)^2; y^2 = 4R_2 r = 4R_2 x.$$

The same equation can be obtained by using limits in (5) or (8):

$$\lim_{R_1 \to \infty} y = \lim_{R_1 \to \infty} \frac{2}{R_1 + R_2}\sqrt{R_1 R_2 (R_1 + R_2 - x) x} = 2\sqrt{R_2 x}.$$

Convention: Throughout the remaining part of the paper the three tangency situations described in Theorems 1–3 will be referred to as Internal, External, and Infinitely large respectively.

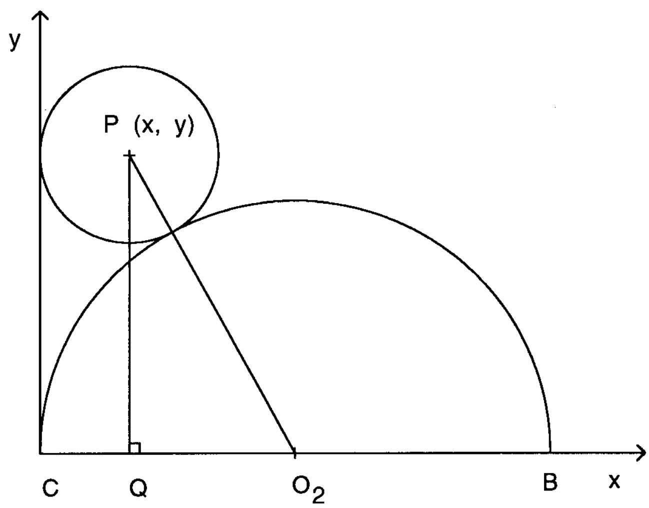

Figure 3.

2 Main Theorem

Now we reconsider the three tangency situations covered in Theorems 1–3 and look at other aspects of the drawn circle using the same conventions.

Theorem 4 *Two circles (or a line and a circle) are given. A circle is drawn to be tangent to both; points of contact being G and H as shown in Figures 4–6.*

(1) *Line GH meets the x-axis at a fixed point.*
(2) *The distance of this point to the point of contact of the given circles is the Geometric Mean of its distances to G and H.*

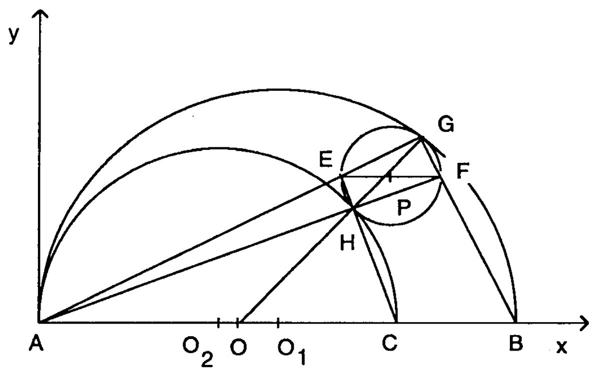

Figure 4.

Proof. Again we supply a proof for the Internal case and give the corresponding results for the other two. We know from (4) that the equation of the conic containing P in the form

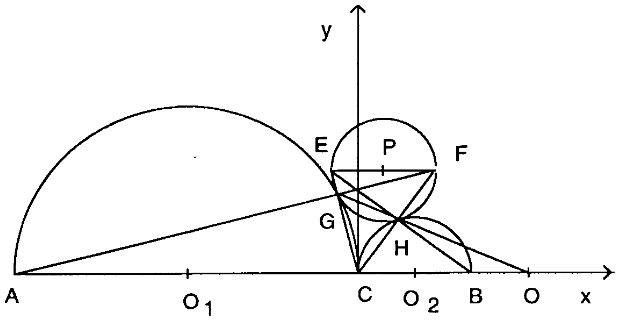

Figure 5.

$y = f(r)$ is

$$y = \frac{2}{R_1 - R_2}\sqrt{R_1 R_2 (R_1 - R_2 - r) r}.$$

If we let $y/2r = \lambda$ and solve this equations simultaneously with (4) we get

$$r = \frac{R_1 R_2 (R_1 - R_2)}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2}. \tag{9}$$

Substitution of (9) into (3) and (4) give:

$$x = \frac{R_1 R_2 (R_1 + R_2)}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2}, \tag{10}$$

$$y = \frac{2\lambda R_1 R_2 (R_1 - R_2)}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2} \tag{11}$$

In Figure 4, $\overline{EF}$ is parallel to the x-axis and is the diameter of the circle with centre $P(x, y)$ and radius r. The coordinates of the endpoints of this diameter $E(x - r, y)$ and $F(x + r, y)$

$$E\left[\frac{2R_1 R_2^2}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2}, \frac{2\lambda R_1 R_2 (R_1 - R_2)}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2}\right],$$

$$F\left[\frac{2R_1^2 R_2}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2}, \frac{2\lambda R_1 R_2 (R_1 - R_2)}{\lambda^2 (R_1 - R_2)^2 + R_1 R_2}\right]$$

Lines AE and BF meet in G and are perpendicular to each other. The equations of these lines are

$$y = \frac{\lambda(R_1 - R_2)}{R_2} x \tag{12}$$

and

$$y = \frac{-R_2}{\lambda(R_1 - R_2)}(x - 2R_1) \tag{13}$$

respectively. On the other hand both of the lines AF and CE go through H meeting at a right angle. Their equations are:

$$y = \frac{\lambda(R_1 - R_2)}{R_1} x \tag{14}$$

and

$$y = \frac{-R_1}{\lambda(R_1 - R_2)}(x - 2R_2) \tag{15}$$

respectively. Solving (12) and (13) simultaneously leads to the coordinates of G, and the common solution of (14) and (15) gives those of H.

$$G\left[\frac{2R_1R_2^2}{\lambda^2(R_1-R_2)^2+R_2^2}, \frac{2\lambda R_1R_2(R_1-R_2)}{\lambda^2(R_1-R_2)^2+R_2^2}\right],$$

$$H\left[\frac{2R_1^2R_2}{\lambda^2(R_1-R_2)^2+R_1^2}, \frac{2\lambda R_1R_2(R_1-R_2)}{\lambda^2(R_1-R_2)^2+R_1^2}\right].$$

Equation of line GH is:

$$y - \frac{2\lambda R_1R_2(R_1-R_2)}{\lambda^2(R_1-R_2)^2+R_2^2} = \frac{\lambda(R_1^2-R_2^2)}{R_1R_2-\lambda^2(R_1-R_2)^2}\left(x - \frac{2R_1R_2^2}{\lambda^2(R_1-R_2)^2+R_2^2}\right). \tag{16}$$

The x-intercept of this line $O\,(2R_1R_2/(R_1+R_2), 0)$. The coordinates of this point are independent of λ.

$$\overline{OH} = \frac{2R_1R_2\sqrt{\left[R_1R_2-\lambda^2(R_1-R_2)^2\right]^2+\lambda^2(R_1^2-R_2^2)^2}}{\left[\lambda^2(R_1-R_2)^2+R_1^2\right](R_1+R_2)},$$

$$\overline{OG} = \frac{2R_1R_2\sqrt{\left[R_1R_2-\lambda^2(R_1-R_2)^2\right]^2+\lambda^2(R_1^2-R_2^2)^2}}{\left[\lambda^2(R_1-R_2)^2+R_2^2\right](R_1+R_2)}.$$

The power of O with respect to the circle with centre P; $\overline{OH}\times\overline{OG} = (2R_1R_2/(R_1+R_2))^2$ The tangent from O to this circle has the constant length $2R_1R_2/(R_1+R_2)$. The radius of the circle with centre O which also goes through A is the Harmonic Mean of R_1 and R_2.

External case:

When the drawn circle is externally tangent to the given circles (see Figure 5) the coordinates of G and H can be worked out using the same approach;

$$G\left[\frac{-2R_1R_2^2}{\lambda^2(R_1+R_2)^2+R_2^2}, \frac{2\lambda R_1R_2(R_1+R_2)}{\lambda^2(R_1+R_2)^2+R_2^2}\right],$$

$$H\left[\frac{2R_1^2R_2}{\lambda^2(R_1+R_2)^2+R_1^2}, \frac{2\lambda R_1R_2(R_1+R_2)}{\lambda^2(R_1+R_2)^2+R_1^2}\right].$$

Equation of line GH is

$$y - \frac{2\lambda R_1R_2(R_1+R_2)}{\lambda^2(R_1+R_2)^2+R_2^2} = \frac{-\lambda(R_1^2-R_2^2)}{R_1R_2+\lambda^2(R_1+R_2)^2}\left(x + \frac{2R_1R_2^2}{\lambda^2(R_1+R_2)^2+R_2^2}\right).$$

The x-intercept of this line $O\,(2R_1R_2/(R_1-R_2), 0)$.

The power of O with respect to the circle with centre P:

$$\overline{OH}\times\overline{OG} = (2R_1R_2/(R_1-R_2))^2.$$

The tangent from O to this circle has the constant length $2R_1R_2/(R_1-R_2)$. The radius of the circle with centre O which also goes through C is the Harmonic Mean of $\overline{OO_1}$ and $\overline{OO_2}$ as $2\overline{OO_1}\times\overline{OO_2}/(\overline{OO_1}+\overline{OO_2}) = 2R_1R_2/(R_1-R_2)$.

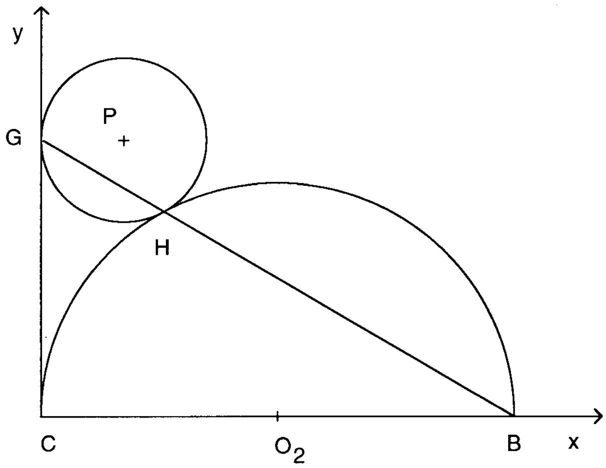

Figure 6.

Infinitely large: (Figure 6)
Now we look at the third case in which we have a line and a circle tangent to it. If one of the circles is infinitely large ($R_1 \to \infty$) as shown in Figure 6, $G\left(0, 2R_2/\lambda\right)$ and $H\left(2R_2/(\lambda^2+1), 2\lambda R_2/(\lambda^2+1)\right)$.

Equation of line GH is $y = (2R_2 - x)/\lambda$ and its x-intercept $B(2R_2, 0)$.

$$\overline{BH} = \frac{2R_2\lambda\sqrt{\lambda^2+1}}{\lambda^2+1}, \quad \overline{BG} = \frac{2R_2\sqrt{\lambda^2+1}}{\lambda}, \quad \overline{BH} \times \overline{BG} = 4R_2^2 = \overline{CB}^2.$$

The tangent from point B to the drawn circle will have a constant length $2R_2$. □

3 Corollaries of Theorem 4

Corollary 1 *In the Internal case the locus of the points of contact of any two tangent circles which are also tangent to the given circles is the circle with centre O and radius $2R_1R_2/(R_1+R_2)$. In the External case the radius of this circle is $2R_1R_2/(R_1-R_2)$. In the third case (Infinitely large) the locus of the points of contact is the circle with centre B and radius $2R_2$.*

Corollary 2 *The line joining the centres of the two tangent circles is tangent to the circle with centre O radius $2R_1R_2/(R_1+R_2)$ (External $2R_1R_2/(R_1-R_2)$, Infinitely large $2R_2$). The point of contact of the two tangent circles is also the point of contact of the line joining their centres and the locus of their point of contact.*

Corollary 3 *The y-intercept of this line is equidistant from the point of contact of these circles and the point of contact of the two given circles.*

4 Infinite Chains of Tangent Circles

The following formulae where derived in [1]. We introduce them here as the rest of the work of the paper is largely based on them.

Internal tangency:
In the infinite chains shown in Figure 7, if the centre of the nth circle is $C_n(x_n, y_n)$ and its

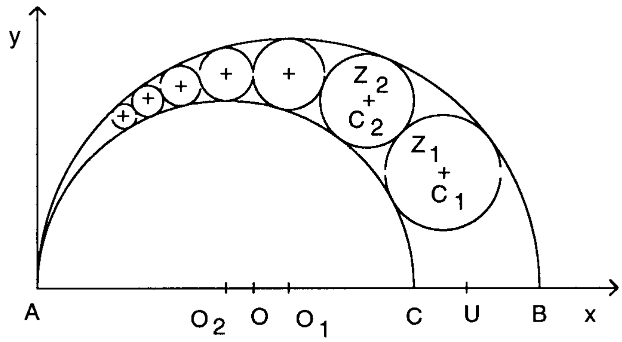

Figure 7.

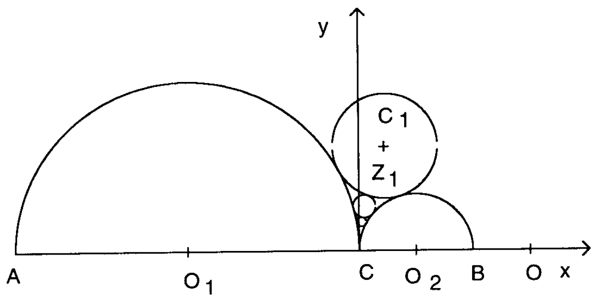

Figure 8.

radius r_n then,

$$r_n = \frac{R_1R_2(R_1 - R_2)}{[(k+n-1)(R_1 - R_2)]^2 + R_1R_2}, \tag{17}$$

$$x_n = \frac{R_1R_2(R_1 + R_2)}{[(k+n-1)(R_1 - R_2)]^2 + R_1R_2}, \tag{18}$$

$$y_n = \frac{2(k+n-1)R_1R_2(R - R_2)}{[(k+n-1)(R_1 - R_2)] + R_1R_2} \quad \text{where} \quad k = \frac{y_1}{2r_1}. \tag{19}$$

External tangency (Figure 8)

$$r_n = \frac{R_1R_2(R_1 + R_2)}{[(k+n-1)(R_1 + R_2)]^2 - R_1R_2},$$

$$x_n = \frac{R_1R_2(R_1 - R_2)}{[(k+n-1)(R_1 + R_2)]^2 - R_1R_2},$$

$$y_n = \frac{2(k+n-1)R_1R_2(R + R_2)}{[(k+n-1)(R_1 + R_2)] - R_1R_2}.$$

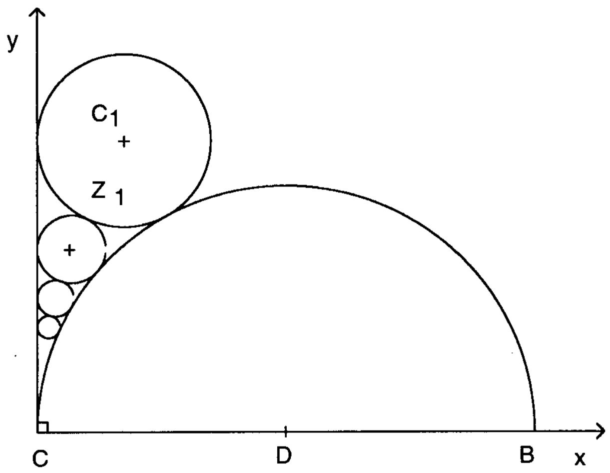

Figure 9.

Infinitely large:
In the chain shown in Figure 9 the coordinates of the centre of the nth circle can also be worked out with limits.

$$r_n = x_n = \frac{R_2}{(k+n-1)^2} \quad \text{and} \quad y_n = \frac{2R_2}{k+n-1} \quad \left(R_2 = \frac{\overline{CB}}{2}\right).$$

5 Coordinates of the Points of Contact

Internal tangency
Let the points of contact of the circles of the chain be $Z_1, Z_2, Z_3, \dots$ (see Figure 7). All these points are on the circle with centre O and radius $2R_1R_2/(R_1+R_2)$. The line joining the centres of any two consecutive circles of the chain is tangent to this circle. The equation of the line through the centres of two neighboring circles e.g. C_n and C_{n+1} is

$$y = \frac{2\left[(k+n)(k+n-1)(R_1-R_2)^2 - R_1R_2\right]}{(2k+2n-1)^2(R_1^2-R_2^2)^2}x + \frac{2R_1R_2}{(2k+2n-1)(R_1-R_2)}. \tag{20}$$

The point of contact of these circles lie on

$$\left(x - \frac{2R_1R_2}{R_1+R_2}\right)^2 + y^2 = \left(\frac{2R_1R_2}{R_1+R_2}\right)^2. \tag{21}$$

The common solution of (20) and (21) produces

$$x = \frac{2R_1R_2(R_1+R_2)}{[2(k+n+1)(k+n-2)+5]\,(R_1-R_2)^2 + 2R_1R_2},$$

$$y = \frac{2(2k+2n-1)R_1R_2(R_1-R_2)}{[2(k+n+1)(k+n-2)+5]\,(R_1-R_2)^2 + 2R_1R_2}.$$

These are the coordinates of Z_n.

External tangency

The points of contact of the circles forming the chain $Z_1, Z_2, Z_3, \ldots$ (see Figure 8) lie on the circle with centre O and radius $2R_1R_2/(R_1 - R_2)$.

The equation of the line through the centres of two consecutive circles. C_n and C_{n+1} is

$$y = \frac{2\left[(k+n)(k+n-1)(R_1+R_2)^2 + R_1R_2\right]}{(2k+2n-1)^2(R_1^2 - R_2^2)^2}x + \frac{2R_1R_2}{(2k+2n-1)(R_1+R_2)}.$$

The point of contact of these circles lie on

$$\left(x - \frac{2R_1R_2}{R_1-R_2}\right)^2 + y^2 = \left(\frac{2R_1R_2}{R_1-R_2}\right)^2.$$

The coordinates of Z_n:

$$x = \frac{2R_1R_2(R_1-R_2)}{[2(k+n+1)(k+n-2)+5](R_1+R_2)^2 - 2R_1R_2},$$

$$y = \frac{2(2k+2n-1)R_1R_2(R_1+R_2)}{[2(k+n+1)(k+n-2)+5](R_1+R_2)^2 - 2R_1R_2}.$$

Infinitely large

In this case the coordinates of Z_n can be obtained by finding the limit for $R_1 \to \infty$ of the coordinates of Z_n in either one of the two previous cases (External or Internal)

$$Z_n\left[\frac{2R_2}{2(k+n+1)(k+n-2)+5}, \frac{2(2k+2n-1)R_2}{2(k+n+1)(k+n-2)+5}\right].$$

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[1] H. Tahir, *Infinite chains of tangent circles*, AustMS Gazette **26** (1999), 128–133.
[2] H. Tahir, *Conic Sections and Tangent Circles*, Melbourne (1999).

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