Quadrance graphs

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Abstract

In this note, we will see an application of **universal geometry** in combinatorics. Let q be any odd prime power of the form $q = 4l + 3$ for some integer l, and F_q be the finite field of order q. The folloing definitions follow the book "Divine porpotions: rational trigeomery to universal geometry" by Wildberger (see [6]).

Definition 1 The **quadrance** $Q(A_1, A_2)$ between the points $A_1 = [x_1, y_1]$, and $A_2 = [x_2, y_2]$ is the number

$$Q(A_1, A_2) := (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Definition 2 A **circle** $C_k(A)$ in a finite field F_q with center $A = [x, y]$ and quadrance $k \in F_q$ is the set of all points X in $F_q \times F_q$ such that

$$Q(A, X) = k.$$

Let us begin the discussion with the following lemma about the number of intersection points between two circles in $F_q \times F_q$.

Lemma 1 *For any $i, j \neq 0$ in F_q. Let X, Y be two distinct points in F_q^2 such that $k = Q(X, Y) \neq 0$. Then the size of the intersections of two circles $C_i(X)$, $C_j(Y)$ only depends on i, j and k. Precisely, let $f(i, j, k) := ij - (k - i - j)^2/4$. Then the number of intersection points is p_{ij}^k, where*

$$p_{ij}^k = \begin{cases} 0 & \text{if } f(i,j,k) \text{ is non-square,} \\ 1 & \text{if } f(i,j,k) = 0, \\ 2 & \text{if } f(i,j,k) \text{ is square.} \end{cases} \tag{1}$$

Proof. Suppose that $X = [m, n]$ and $Y = [m + x, n + y]$ for some $m, n, x, y \in F_q$ then $x^2 + y^2 = k$. Suppose that $Z \in C_i(X) \cap C_j(Y)$ where $Z = [m + x + u, n + y + v]$ for some $u, v \in F_q$. Then we have $u^2 + v^2 = j$ and $(x + u)^2 + (y + v^2) = i$. This implies that $xu + yv = (i - j - k)/2$. But we have $(xu + yv)^2 + (xv - yu)^2 = (x^2 + y^2)(u^2 + v^2)$ so

$$(xv - yu)^2 = kj - \frac{(i - j - k)^2}{4} = ij - \frac{(k - i - j)^2}{4} = f(i, j, k).$$

If $f(i, j, k)$ is a non-square number in F_q then it is clear that there does not exist such x, y, u, v, or $p_{ij}^k = 0$. Otherwise, let $\alpha = (i - j - k)/2$ and $f(i, j, k) = \beta^2$ for some $0 \leq \beta \leq (p + 1)/2$ then

$$xv - yu = \pm\beta, \qquad xu + yv = \alpha.$$

Solving for (u, v) with respect to (x, y) we have

$$u = (\alpha x \mp \beta y)/k, \qquad v = (\alpha y \pm \beta x)/k.$$

If $\beta = 0$ then we have only one (u, v) for each (x, y), but if $\beta \neq 0$ then we have two pairs (u, v). This implies (1) and concludes the proof of the theorem. □

The **quadrance graph** V_q is defined on the vertex set $F_q \times F_q$. The pair (X, Y) with $X, Y \in F_q \times F_q$ is an edge of V_q if and only if $Q(X, Y)$ is a (nonzero) square in F_q. We will first show that V_3 gives us a critial colouring for Ramsey number $R(K_4 - e, K_4 - e)$; recall that $K_4 - e$ is the complete graph with 4 vertices with one edge deleted, and the Ramsey number $R(K_4 - e, K_4 - e)$ is the minimal number n such that for any 2-colouring of edges of K_n, we can find a monochromatic subgraph $K_4 - e$.

Example We will show that neither V_3 nor $\overline{V_3}$ contains a $K_4 - e$. Suppose that V_3 contains a $K_4 - e$ then there exists 4 points X, Y, Z, T in $F_3 \times F_3$ such that

$$Q(X, Y) = Q(X, Z) = Q(X, T) = Q(Y, Z) = Q(Y, T) = 1.$$

It implies that the circle $C_1(X)$ intersects $C_1(Y)$ at Z and T. We have $f(1, 1, 1) = 0$ in F_3. From Lemma 1, for any X, Y with $Q(X, Y) = 1$, the circle $C_1(X)$ intersects $C_1(Y)$ at exactly one point. It implies that $Z \equiv T$, which is a contradiction. Thus, V_3 does not contains $K_4 - e$. Similarly, $f(2, 2, 2) = 0$ in F_3 so $\overline{V_3}$ does not have a subgraph $K_4 - e$. Therefore, $R(K_4 - e, K_4 - e) \geq 10$ since our graph has 9 vertices. In [3], we know that $R(K_4 - e, K_4 - e) = 10$. It has been verified (by computer) that there is only one (up to graph isomorphism) 2-colouring of K_9 with no monochromatic subgraph $K_4 - e$. Perhaps the construction here is the most intuitive one.

Using Lemma 1, we can show that the graph V_q is a strongly regular graph with parameters $\{q^2, (q^2-1)/2, (q^2-5)/4, (q^2-1)/4\}$: V_q is $(q^2-1)/2$-regular, any two adjacent vertices have $(q^2-5)/4$ common neighbours and any two non-adjacent vertices have $(q^2-1)/4$ common neighbours. The basic idea is for a fixed $k \in F_q$ we need to count the number of pair $(i^2, j^2) \in F_q \times F_q$ such that $f(i^2, j^2, k)$ is a nonzero square in F_q. The direct proof, however, is lengthy and technical and will appear elsewhere.

Note that the well-known Paley graph is defined in a similar way. The Paley graph is defined on a vertex set F_q for some odd prime power g of the form $g = 4l + 1$. The pair (i, j) with $i, j \in F_g$ is an edge of P_g if and only if $i - j$ is a nonzero square in F_g. When $g \equiv 1 \pmod 4$ then -1 is square in F_g so the Paley graph is well-defined. We know that (see [1], page 315-323) the Paley graph P_g is also a strongly regular graph with parameters $\{g, (g-1)/2, (g-5)/4, (g-1)/4\}$. Thus, the quadrance graph V_q has the same parameters with the Paley graph P_{q^2} for all prime power q of the form $q = 4l + 3$. A natural question is whether the two graphs are isomorphic for all prime power $q \equiv 3 \pmod 4$. Interestingly, the answer is affirmative.

Theorem 1 *Let $q \equiv 3$ (mod 4) be a prime power. Then the quadrance graph V_q is isomorphic to the Paley graph P_{q^2}.*

Proof. Let $f(x) = x^2 + 1 \in F_q[x]$ and α be a root of $f(x)$. Since $q \equiv 3 \pmod 4$, we have $f(x)$ is irreducible over F_q. Thus, each element $\beta \in F_{q^2}$ can be uniquely represented in the form

$$\beta = x_\beta + y_\beta \alpha \qquad \text{with} \qquad x_\beta, y_\beta \in F_q. \tag{2}$$

Let $X_\beta = [x_\beta, y_\beta] \in F_q \times F_q$ for $\beta \in F_{q^2}$. We show that the isomorphism which maps $\beta = x_\beta + y_\beta \alpha \in F_{q^2}$ to $[x_\beta, y_\beta] \in F_q \times F_q$ induces a graph isomorphism between the Paley graph P_{q^2} and the quadrance graph V_q. Both P_{q^2} and V_q are $(q^2-1)/2$-regular so it suffices to show that (β, γ) is an edge of P_{q^2} then (X_β, X_γ) is an edge of V_q.

Suppose that (β, γ) is an edge of P_{q^2} then $\beta - \gamma = (x_\beta - x_\gamma) + (y_\beta - y_\gamma)\alpha$ is square in F_{q^2}. Then there exists $\rho, \tau \in F_q$ such that $\beta - \gamma = (\rho + \tau\alpha)^2$. From (2), we have

$$\begin{aligned} x_\beta - x_\gamma &= \rho^2 + \tau^2\alpha^2 = \rho^2 - \tau^2, \\ y_\beta - y_\gamma &= 2\rho\tau. \end{aligned}$$

This implies that

$$Q(X_\beta, X_\gamma) = (\rho^2 - \tau^2)^2 + 4\rho^2\tau^2 = (\rho^2 + \tau^2)^2 \qquad \text{is square in } F_q.$$

Hence (X_β, X_γ) is an edge of V_q. This concludes the proof. □

Thus, we can deduce that the graph V_q is a strongly regular graph with parameters $\{q^2, (q^2-1)/2, (q^2-5)/4, (q^2-1)/4\}$ from Theorem 1 and the standard fact of the Paley graph. Furthermore, we can also show that the quadrance graph V_q is a special case of the graph defined in Example (2.21) in [4].

References

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