# The inheritance problem and monotone systems

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## 1 The inheritance problem and its mathematical formulation

*To divide an inheritance three brothers turn to a judge. Secretly however, each of them bribes the judge. What a given brother inherits depends continuously and monotonically on the bribes: it is monotone increasing in his own bribe and it is monotone decreasing in everybody else's bribe. Show that if the eldest brother does not give too much to the judge, then the others can choose their bribes so that the decision will be fair, i.e., each of them gets equal share.*

To deal with problems like this, one needs to translate them into the language of mathematics. We have to get to the bare bone of the statement without such expressions as "secretly, however" or "eldest brother".

Let the brothers be I, II and III, the last being the eldest. To mathematically formulate the problem first of all we do not deal with real money (dollars or pennies), and we assume that a bribe (or a share) can be any nonnegative number. Thus, the bribe of I is some number $x_1$, and likewise let $x_2$ and $x_3$ be the bribes of brothers II and III. When talking about the share of each brother we only have to specify the proportion that each of them gets from the total inheritance. Let this proportion for I be $g_1$. Of course, this proportion depends on the bribes $x_1, x_2, x_3$, so we are dealing here with a function $g_1(x_1, x_2, x_3)$ of three variables. Likewise, the proportion that the second and third brothers get are some functions $g_2(x_1, x_2, x_3)$ and $g_3(x_1, x_2, x_3)$, and at this moment there is no reason to assume that these functions are the same (actually, we shall see that they are not!), or one of them can be obtained from another by permuting $x_1, x_2$ ad $x_3$.

The "continuity" in the problem translates as the continuity of the functions $g_1, g_2, g_3$ of three variables (which is not the same as continuity separately in each variable!). The monotonicity condition is that $g_1$ increases as $x_1$ increases, and decreases as $x_2$ or $x_3$ increases (which are the bribes of the other brothers). Likewise, $g_2$ resp. $g_3$ are increasing in the variable $x_2$ resp. $x_3$ and decreases in every other variable. Here and in what follows "increasing" and "decreasing" means strict increase and decrease.

It is implicitly assumed that originally the judge was impartial, which means that if nobody gives any bribe, then each brother gets 1/3 of the heritage, i.e. $g_j(0,0,0) = 1/3$ for all $j = 1, 2, 3$. Finally, the conclusion is that if $x_3$, the bribe of the eldest brother, is not too large, then $x_1$ and $x_2$ can be chosen so that the decision will be fair again. This means that there is a number, say $\alpha > 0$, such that if $0 \le x_3 \le \alpha$, then there are $x_1, x_2 \ge 0$ so that $g_j(x_1, x_2, x_3) = 1/3$ for all $j$, but it will be more convenient to rewrite this in the form $g_j(x_1, x_2, x_3) = g_j(0,0,0)$ for all $j$.

One final word: the bribes should not be arbitrary large numbers, so we shall assume that they are all restricted to the range $0 \le x_j \le a$ with some number $a > 0$. Then $[0, a]^3$ is the domain of the functions $g_1, g_2, g_3$.

In summary, we have for these three functions:

(A) $g_1, g_2, g_3$ *are continuous functions on a cube* $[0, a]^3$,

(B) $g_j(x_1, x_2, x_3)$ *is strictly monotone increasing in* $x_j$ *and strictly monotone decreasing in every* $x_i$ *with* $i \neq j$ *and*

(C) $g_1(x_1, x_2, x_3) + g_2(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)$ *is constant.*

Actually, in this example the constant in (C) is 1, and property (C) expresses the fact that the three brothers get the whole inheritance.

This is the mathematical model we had in mind. Anything we prove in connection with the inheritance problem that uses only properties (A)–(C) will remain valid in other situations the mathematical formulation of which is given by three functions with properties (A)–(C). Below we shall list several systems that have properties (A)–(C), and from now on we shall call each such system a *monotone system*.

## 2 The case of more than three brothers

The problem is not restricted to three brothers, in fact, the same formulation holds for any number of brothers :

*To divide an inheritance n brothers turn to a judge. Secretly however, . . .*

This was one of the problems of the 1991 Miklós Schweitzer Mathematical Contest[1] organized by the János Bolyai Mathematical Society in Hungary (see [3, p. 46]).

In this case we have $n$ bribes, say $x_1, \dots, x_n$, and if the proportion that the $j$-th brother gets when the bribes are $x_1, \dots, x_n$ is $g_j(x_1, \dots, x_n)$, then these $n$ functions $g_1(x_1, \dots, x_n)$, $\dots, g_n(x_1, \dots, x_n)$ of $n$ variables satisfy:

(A) $g_j$ *is a continuous function on a cube* $[0, a]^n$,

(B) $g_j(x_1, \dots, x_n)$ *is strictly monotone increasing in* $x_j$ *and strictly monotone decreasing in every* $x_i$ *with* $i \neq j$ *and*

(C) $\sum_{j=1}^{n} g_j(x_1, \dots, x_n)$ *is constant.*

The claim is that there is an $\alpha > 0$ with the property that if $0 \leq x_n \leq \alpha$, then there are $x_1, \dots, x_{n-1} \in [0, a]$ such that each $g_j(x_1, \dots, x_n)$ equals $1/n$. Since the judge was originally impartial, the same is true of the values $g_j(\underline{\mathbf{0}}) = g_j(0, \dots, 0)$, therefore the problem can again be reformulated by saying that if $0 \leq x_n \leq \alpha$, then there are $x_1, \dots, x_{n-1} \in [0, a]$ such that $g_j(x_1, \dots, x_n) = g_j(\underline{\mathbf{0}})$ for all $j$. In this formulation it is irrelevant if originally the judge is impartial of biased, $g_j(x_1, \dots, x_n) = g_j(\underline{\mathbf{0}})$ can be achieved provided $x_n$ is not too large.

In what follows we shall call any system $g_1(\underline{\mathbf{x}}), \dots, g_n(\underline{\mathbf{x}})$, $\underline{\mathbf{x}} = (x_1, \dots, x_n)$ with the properties (A),(B) and (C) a *monotone system*.

Although in the inheritance problem all the functions $g_j$ are nonnegative, this will not be required in what follows when we talk about monotone systems.

The clause "If the eldest brother does not give too much to the judge" in the conclusion is important, i.e. for every monotone system there is an $\alpha > 0$ such that the conclusion holds if $0 \leq x_n \leq \alpha$. It may not be true for larger values of $x_n$ (i.e. if the eldest brother gives more than $\alpha$). Here is a simple example (for the original $n = 3$ case) showing that if $\beta > 0$ is given, and the eldest brother gives at least $\beta$ bribe, then fair decision cannot be achieved: let

$$g_1(x_1, x_2, x_3) = \frac{1}{3} + x_1 + \frac{\beta x_1}{x_1 + 1} - x_2 - x_3,$$

[1] This is a unique mathematical contest for university students organized since 1949. There are no age groups; each year about ten problems are proposed for ten days, and the students can use any literature they want. Accordingly, the problems are more challenging than on usual sit-in competitions. The problems and solutions from the period 1949–1961 can be found in [2], and from the period 1962–1992 in [3].

$$g_2(x_1,x_2,x_3)=\frac{1}{3}+x_2+\frac{\beta x_2}{x_2+1}-x_1-x_3$$

and

$$g_3(x_1,x_2,x_3)=\frac{1}{3}+2x_3-\frac{\beta x_1}{x_1+1}-\frac{\beta x_2}{x_2+1}.$$

In fact, properties (A)–(C) are easily established, but if $x_3 \geq \beta$, then, since $\frac{x_1}{x_1+1}<1$ and $\frac{x_2}{x_2+1}<1$, we have

$$g_3(x_1,x_2,x_3)\geq\frac{1}{3}+2\beta-\frac{\beta x_1}{x_1+1}-\frac{\beta x_2}{x_2+1}>\frac{1}{3},$$

i.e. the eldest brother gets more than 1/3 of the total inheritance regardless of the values $x_1, x_2$.

A similar example, namely

$$g_j(x_1,...,x_n)=\frac{1}{n}+(n-2)x_j+\frac{\beta x_j}{x_j+1}-x_1-\cdots-x_{j-1}-x_{j+1}-\cdots-x_n$$

if $j=1,2,...,n-1$, and

$$g_n(x_1,...,x_n)=\frac{1}{n}+(n-1)x_n-\frac{\beta x_1}{x_1+1}-\frac{\beta x_2}{x_2+1}-\cdots-\frac{\beta x_{n-1}}{x_{n-1}+1},$$

works for the case of $n$ brothers.

It is also easy to show that in condition (B) the strict monotonicity cannot be relaxed to non-strict monotonicity.

## 3 Some monotone systems

In this section we list a few monotone systems. These examples show that monotone systems appear in quite different situations. As is the case with any properly formulated mathematical models, the conclusions one draws from the assumptions in the model are valid in every concrete case that follows the model. Thus, in each of the following examples the equal share conclusion $g_j(x_1,\ldots,x_n)=g_j(\underline{\mathbf{0}})$, $j=1,2,\ldots,n$ is true (but has a different meaning), because it will be proven in the next section solely from the assumptions (A)–(C) of monotone systems.

The following observation helps in verifying the monotonicity property (B). Since the sum of the functions is constant, for the monotonicity property it is enough to require that each $g_i$ is strictly monotone decreasing in every $x_j$ with $j \neq i$. In fact, then if $x_j$ increases then every other $g_i$ decreases, and so does $\sum_{i\neq j} g_i$. But then $g_j = 1-\sum_{i\neq j} g_i$ increases, and this proves that the increase of $g_j$ in $x_j$ automatically follows.

### Market share

Suppose that $n$ companies are producing the same product. Let $x_j$ be the investment of the $j$-th company and $g_j(x_1,\ldots,x_n)$ its market share. It is reasonable to assume that if another, say $i$-th company with $i \neq j$, increases its investment, then the market share of the $j$-th company decreases, thus we have a monotone system.

### System of connected pipes

Suppose that we have a system of $n$ connected pipes (see Figure 1), $x_j$ is the diameter of the $j$-th pipe and $g_j(\underline{\mathbf{x}})$ is the amount of liquid in the $j$-th pipe. If $x_i$ with $i \neq j$ increases, then the liquid level decreases, and so is the amount of the liquid in the $j$-th pipe, i.e. $g_j(\underline{\mathbf{x}})$ decreases. Therefore, this is a monotone system.

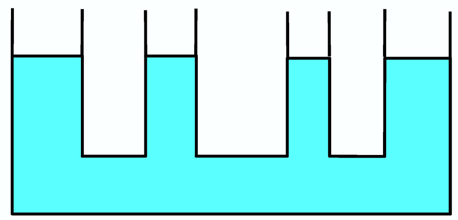

Figure 1. System of connected pipes

**Resistors in series**

Let the resistors $R_1, \ldots, R_n$ be connected in series in a circuit (see Figure 2), let $x_j$ be the resistance of $R_j$ and $g_j(\underline{\mathbf{x}})$ the potential drop on $R_j$. If $x_i$ increases then the cumulative resistance $R = x_1 + \cdots + x_n$ increases, hence the current $I = U/R$ decreases. Now if $i \neq j$, then this implies that the potential drop $x_j I$ on $R_j$ decreases, hence we have a monotone system.

**Capacitors in parallel**

Let the capacitors $C_1, \ldots, C_n$ be connected in parallel in a circuit, let $x_j$ be the capacity of $C_j$ and $g_j(\underline{\mathbf{x}})$ the total charge on $C_j$. If $x_i$ increases then the cumulative capacity $C = x_1 + \cdots x_n$ increases, hence the potential $U = Q/C$ decreases. Now for $i \neq j$ this implies that the charge $x_j U$ on $C_j$ decreases, and we have again a monotone system.

**Conservation laws of physics**

Physics is abundant in monotone systems; conservation laws easily give rise to such systems. Consider for example a rotating disk on which there are $n$ point masses $m_1, \ldots, m_n$. Let $x_j$ be the distance of $m_j$ from the center of rotation. The angular momentum $\sum_j m_j x_j^2 \omega$, where $\omega$ is the angular speed, is invariant, thus if $x_i$ increases, then $\omega$ must decrease. Hence, if $g_j(\underline{\mathbf{x}}) = m_j x_j^2 \omega$ denotes the angular momentum of the $j$-th mass, then for $j \neq i$ this will decrease, and so the system $\{g_j(\underline{\mathbf{x}})\}$, $j = 1, \ldots, n$ is a monotone system.

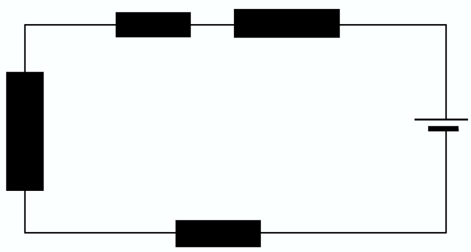

Figure 2. Resistors in series

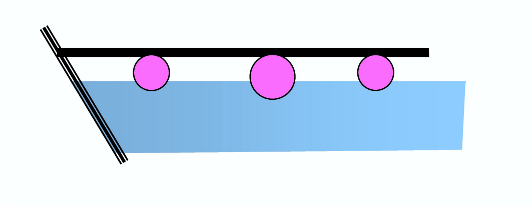

Figure 3. Supporting a pontoon bridge

### Supporting a pontoon bridge

If $n$ pontoons support a bridge (c.f. Figure 3), $x_j$ is the size of the $j$-th pontoon and $g_j(\underline{\mathbf{x}})$ is the weight force on it, then it is easy to see that we have a monotone system.

### Scheduling jobs on a mainframe computer

Suppose $n$ jobs with priorities $x_1, \ldots, x_n$ are to be run on a computer, and let $g_j(\underline{\mathbf{x}})$ be the time needed to finish the $j$-th job. If $x_i$ increases then for $j \neq i$ the time $g_j(\underline{\mathbf{x}})$ increases, so in this case $\{-g_j(\underline{\mathbf{x}})\}_{j=1}^n$ will form a monotone system.

### Arteries and organs

Suppose that $n$ arteries supply blood to $n$ organs. Let $x_j$ be the diameter of the $j$-th artery and $g_j(\underline{\mathbf{x}})$ the amount of blood reaching the $j$-th organ (say in unit time). If $x_i$ increases then blood pressure decreases, hence in unit time less blood will flow through the $j$-th artery for $j \neq i$.

### Equilibrium measures

Suppose that we put a unit charge on a conductor $E$ consisting of $n$ plates $E_1, \ldots, E_n$. The charge can move freely on the conductor, and it will distribute itself in a way that minimizes its inner energy. Let $\mu_E$ denote this so called equilibrium distribution. What happens if one of the plates, say the $i$-th one shrinks, e.g. we cut off a part $K$ of it (c.f. Figure 4, where the plates are the triangle, the quadruple and the hexagon, and we cut off one corner of the triangle)? To maintain equilibrium, the charge sitting originally on $K$ has to move to the rest of the conductor (on Figure 4 this is indicated by the arrows), which means that the total charge on a different plate $E_j$, $j \neq i$ will increase. Thus, if we measure the size of the $j$-th plate with some parameter $x_j$, and $g_j(\underline{\mathbf{x}}) = \mu_E(E_j)$ denotes the total amount of the equilibrium mass on $E_j$, then we get a monotone system.

This example is actually closely related to the 4-th example above on capacitors in parallel.

## 4 Solution to the inheritance problem

We shall use induction to solve the inheritance problem. This is one of the most basic reasonings in mathematics. The induction not only proves the general $n$-brother problem, but it also needs this formulation, for it goes like this: verify the statement for $n = 1$ (which is the empty case, for there is only one brother, and the decision is always fair regardless how much bribe he gives), then using this verify it to $n = 2$, then using this case verify it to $n = 3$, etc. The original problem stops at $n = 3$, but the general case is obtained practically free, for we will prove the induction steps $1 \to 2$, $2 \to 3$, $3 \to 4$ in a unified manner (and

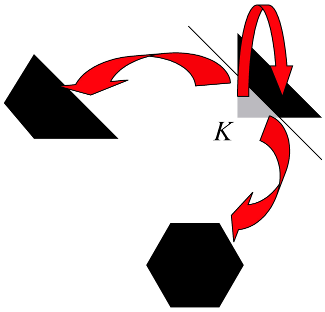


Figure 4. Moving charge

this is the heart of proof by induction), and it takes the same effort to prove the case $n = 3$ from the case $n = 2$ as to prove the general case of $n$ brothers from that of $n - 1$ brothers.

We can consider $g_j(\underline{\mathbf{x}}) - g_j(\underline{\mathbf{0}})$ instead of $g_j(\underline{\mathbf{x}})$ (these also have properties (A)–(C)), hence we may suppose $g_j(\underline{\mathbf{0}}) = 0$ for all $j$. Then property (C) takes the form

$$\sum_{j=1}^{n} g_j(\underline{\mathbf{x}}) = 0. \tag{1}$$

We have to show that there exists $\alpha > 0$ such that for every $0 \le x_n \le \alpha$ there are values $x_1, \ldots, x_n$ with $g_j(x_1, \ldots, x_n) = 0$ for all $j$. Here each of $x_1, \ldots, x_{n-1}$ may (and generally will) depend on the choice of $x_n$, so we write $x_1 = x_1(x_n), \ldots, x_{n-1} = x_{n-1}(x_n)$ to show this dependence.

If $n = 1$, there is nothing to prove. Assume that the assertion holds for $n - 1$ functions. We claim that there is a number $b > 0$ such that if $0 \le x_2, \ldots, x_n \le b$ are arbitrary, then there exists one and only one $y = y(x_2, \ldots, x_n)$ satisfying $g_1(y, x_2, \ldots, x_n) = 0$, and this $y$ is a continuous, strictly increasing function of the variables $x_2, \ldots, x_n$. In fact, the uniqueness of $y$ follows from $g_1$ being strictly increasing in its first variable. If $x_2 = x_3 = \ldots = x_n = 0$ then we choose $y = 0$, while for other values the existence of $y$ can be seen as follows. Since $g_1(a, 0, \ldots, 0) > 0$, continuity yields the existence of a $b > 0$ such that for $0 \le x_2, \ldots, x_n \le b$ we have

$$g_1(a, x_2, \ldots, x_n) > 0.$$

On the other hand, property (B) shows

$$g_1(0, x_2, \ldots, x_n) < 0$$

(note not all of $x_2, \ldots, x_n$ is zero), hence, again by continuity, $y$ must exist. Here, and in what follows, we use the fact that continuous functions have the intermediate value property, i.e. if $f$ is a continuous function on an interval, it assumes the values $R < S$ there (i.e. $R = f(r)$ and $S = f(s)$ for some $r, s$) and $R < T < S$, then it also assumes the value $T$ (i.e. $T = f(t)$ for some $t$).

Next we show the monotonicity of $y(x_2, \ldots, x_n)$. If $x'_j > x_j$, $j \ne 1$, then

$$\begin{aligned} g_1(y(x_2, \ldots, x'_j, \ldots, x_n), x_2, \ldots, x'_j, \ldots, x_n) = 0 \\ = \quad g_1(y(x_2, \ldots, x_j, \ldots, x_n), x_2, \ldots, x_j, \ldots, x_n) \\ > \quad g_1(y(x_2, \ldots, x_j, \ldots, x_n), x_2, \ldots, x'_j, \ldots, x_n), \end{aligned}$$

so we must have

$$y(x_2, \ldots, x'_j, \ldots, x_n) > y(x_2, \ldots, x_j, \ldots, x_n)$$

since $g_1$ is monotone increasing in its first coordinate. This shows that indeed, $y(x_2, \ldots, x_j, \ldots, x_n)$ is an increasing function of $x_j$.

Finally, we verify the continuity of $y$. Let $\eta > 0$ be arbitrary. Since

$$g_1(y(x_2, \ldots, x_n) - \eta, x_2, \ldots, x_n) < 0 < g_1(y(x_2, \ldots, x_n) + \eta, x_2, \ldots, x_n),$$

there is a $\delta > 0$ such that for $|x_j - x'_j| \le \delta$, $j = 2, 3, \ldots, n$, we have

$$g_1(y(x_2, \ldots, x_n) - \eta, x'_2, \ldots, x'_n) < 0 < g_1(y(x_2, \ldots, x_n) + \eta, x'_2, \ldots, x'_n).$$

Thus, for $|x_j - x'_j| \le \delta$ we must have (use again that $g_1$ is increasing in $x_1$)

$$y(x_2, \ldots, x_n) - \eta < y(x'_2, \ldots, x'_n) < y(x_2, \ldots, x_n) + \eta,$$

and this is the stated continuity.

After these preparations we make the induction step. Consider the $n - 1$ functions

$$h_j(x_2, \ldots, x_n) = g_j(y(x_2, \ldots, x_n), x_2, \ldots, x_n), \quad j = 2, \ldots, n$$

defined on $[0, b]^{n-1}$ (where $b$ is the constant used during the preparation). They are continuous, they have sum 0 (recall (1) and the fact that by the choice of $y$ we have $g_1(y(x_2, \ldots, x_n), x_2, \ldots, x_n) = 0$), and satisfy the relations $h_j(0, \ldots, 0) = 0$ for all $j$. They are also strictly monotone in the required sense: if $i \neq j$ and $x_i$ increases, then so does $y(x_2, \ldots, x_n)$, and therefore $h_j$ decreases because $g_j(x_1, \ldots, x_n)$ is a decreasing function in both $x_1$ and $x_i$. Thus, $h_2(x_2, \ldots, x_n), \ldots, h_2(x_2, \ldots, x_n)$ form a monotone system of $(n - 1)$ functions. By the induction hypothesis there exists $0 < \alpha' < b$ such that for any $0 \le x_n \le \alpha'$ there are $x_2 = x_2(x_n), \ldots, x_{n-1} = x_{n-1}(x_n)$ in the interval $[0, b]$ satisfying $h_j(x_2, \ldots, x_n) = 0$ for all $j \ge 2$. Now with the values

$$x_1 = x_1(x_n) := y(x_2(x_n), \ldots, x_{n-1}(x_n), x_n),$$

$$x_2 = x_2(x_n), \ldots, x_{n-1} = x_{n-1}(x_n),$$

we have $g_j(x_1, \ldots, x_n) = 0$ for all $j = 1 \ldots, n$ (for $j > 1$ this follows from the corresponding relation for the $h_j$'s, while for $j = 1$ this is a consequence of the choice of $y$).

As this verifies the induction step, the proof is complete.

## 5 Rational decisions and the open mapping property

It turns out that the following variant of the inheritance problem has important applications in mathematical analysis (see e.g. [1],[5] or [6] where the rationality of the equilibrium measure carried by each plates in the last example of Section 3 is of fundamental importance). In it the judge may be biased, the decision still will be "rational".

*No matter what the eldest brother gives to the judge, the others can choose arbitrarily small bribes so that the decision will be "rational" in the sense that each brother gets a rational fraction of the heritage.*

Let $\mathbf{G}(\underline{\mathbf{x}}) = (g_1(\underline{\mathbf{x}}), g_2(\underline{\mathbf{x}}), g_3(\underline{\mathbf{x}}))$, $\underline{\mathbf{x}} = (x_1, x_2, x_3)$ be our monotone system, and recall that we have $g_1(\underline{\mathbf{x}}) + g_2(\underline{\mathbf{x}}) + g_3(\underline{\mathbf{x}}) \equiv 1$. Now the rational decision problem asks us to show that for any fixed $x_3$ there are as small $x_1, x_2$ as we wish such that with $\underline{\mathbf{x}} = (x_1, x_2, x_3)$ each of the values $g_1(\underline{\mathbf{x}}), g_2(\underline{\mathbf{x}})$, $g_3(\underline{\mathbf{x}})$ is a rational number. Note that this is not automatic, since the point $(g_1(\underline{\mathbf{x}}), g_2(\underline{\mathbf{x}}), g_3(\underline{\mathbf{x}}))$ in three-space lies on the plane

$$L = \{(y_1, y_2, y_3)|\ y_1 + y_2 + y_3 = 1\}.$$

But note also that for $g_3(x)$ the rationality automatically follows once we know it for $g_1(\underline{\mathbf{x}})$ and $g_2(\underline{\mathbf{x}})$, for $g_3(\underline{\mathbf{x}}) = 1 - g_1(\underline{\mathbf{x}}) - g_2(\underline{\mathbf{x}})$. Thus, since $x_3$ is fixed, we have to consider now only two functions $g_1(x_1, x_2, x_3)$ and $g_2(x_1, x_2, x_3)$, and in these two functions only two variables, namely $x_1$ and $x_2$ can change.

For fixed $x_3$ therefore we consider the (so called vector valued) function

$$G(x_1, x_2) = (G_1(x_1, x_2), G_2(x_1, x_2)) = (g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3))$$

which maps from (a subset of) the plane into the plane. Suppose that the $g_j$'s have continuous partial derivatives, and consider the Jacobian

$$\mathcal{J} = \begin{vmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_2}{\partial x_1} \\ \frac{\partial G_1}{\partial x_2} & \frac{\partial G_2}{\partial x_2} \end{vmatrix}.$$

The monotonicity properties give that here the diagonal elements are nonnegative and the off-diagonal elements are non-positive. Furthermore, since

$$g_1(\underline{\mathbf{x}}) + g_2(\underline{\mathbf{x}}) + g_3(\underline{\mathbf{x}}) \equiv 1$$

we also have

$$\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_1} + \frac{\partial g_3}{\partial x_1} = 0$$

(the derivative of a constant function is 0), therefore

$$\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_1} = -\frac{\partial g_3}{\partial x_1} \geq 0,$$

again by the monotonicity property (B). It follows, that

$$\frac{\partial G_1}{\partial x_1} \geq -\frac{\partial G_2}{\partial x_1}, \tag{2}$$

and similar reasoning gives

$$\frac{\partial G_2}{\partial x_2} \geq -\frac{\partial G_1}{\partial x_2}, \tag{3}$$

and hence the determinant

$$\mathcal{J} = \frac{\partial G_1}{\partial x_1} \cdot \frac{\partial G_2}{\partial x_2} - \frac{\partial G_1}{\partial x_2} \cdot \frac{\partial G_2}{\partial x_1}$$

is nonnegative. If it is actually positive (which is the case if there is strict inequality in (2) and (3)), then it follows that the Jacobian of the mapping $(x_1, x_2) \to (G_1(x_1, x_2), G_1(x_1, x_2))$ is non-zero. Now there is a basic result in mathematical analysis, called the inverse mapping theorem ([7, Theorem 9.24]), which claims that if the Jacobian of a (continuously differentiable) mapping is not zero at some point $(x_1^*, x_2^*)$, then there is a neighborhood $U$ of $(x_1^*, x_2^*)$ (i.e. an open set $U$ containing the point $(x_1^*, x_2^*)$) and a neighborhood $V$ of the image point $(G_1(x_1^*, x_2^*), G_1(x_1^*, x_2^*))$ such that $(x_1, x_2) \to (G_1(x_1, x_2), G_1(x_1, x_2))$ is a one-to-one mapping between $U$ and $V$ (recall, that a subset $E$ of $\mathbb{R}^2$ is open if with every point $P$ in $E$ there is a disk around $P$ which is also contained in $E$). In particular, if the Jacobian is strictly positive, then $(x_1, x_2) \to (G_1(x_1, x_2), G_1(x_1, x_2))$ maps $(0, a)^2$ onto an open subset of $\mathbb{R}^2$. But then clearly, there are points with rational coordinates in the image set (in every disk there are points with rational coordinates), i.e. there are $x_1, x_2$ (actually, they are dense in $(0, a)$) such that $G_1(x_1, x_2)$ and $G_2(x_1, x_2))$ are both rational, and this is precisely the rational decision problem.

The property that the image of $(0, a)^2$ under

$$(x_1, x_2) \to (G_1(x_1, x_2), G_1(x_1, x_2)) \tag{4}$$

is an open subset of $\mathbb{R}^2$ actually holds for all monotone systems (without any assumptions on partial derivatives). In fact, we show that $(x_1, x_2) \to (G_1(x_1, x_2), G_1(x_1, x_2))$ is a so called homeomorphism, i.e. it is a continuous and one-to-one mapping (it maps different points into different points). There is a deep theorem in topology, called "invariance of domains", which says that the image of an open set under a continuous one-to-one mapping is open (see e.g. [4, Proposition 7.4, p. 79]).

That $(x_1, x_2) \to (G_1(x_1, x_2), G_1(x_1, x_2))$ is continuous follows from assumption (A) for monotone systems. Thus, it is left to show that if $(x_1, x_2) \neq (x_1', x_2')$, i.e. either $x_1 \neq x_1'$ or $x_2 \neq x_2'$, then $(G_1(x_1, x_2), G_1(x_1, x_2)) \neq (G_1(x_1', x_2'), G_1(x_1', x_2'))$. By interchanging the role of $(x_1, x_2)$ and $(x_1', x_2')$ we may assume that $x_1 \geq x_1'$. If $x_2 < x_2'$, then using the monotonicity property (B) as well as property (C) (where now the constant is 1), we can write

$$\begin{aligned} G_1(x_1, x_2) &= g_1(x_1, x_2, x_3) > g_1(x_1, x_2', x_3) \\ &= 1 - g_2(x_1, x_2', x_3) - g_3(x_1, x_2', x_3) \\ &\geq 1 - g_2(x_1', x_2', x_3) - g_3(x_1', x_2', x_3) \\ &= g_1(x_1', x_2', x_3) = G_1(x_1', x_2'). \end{aligned}$$

On the other hand, if $x_2 \geq x_2'$, then

$$\begin{aligned} G_1(x_1, x_2) + G_1(x_1, x_2) &= 1 - g_3(x_1, x_2, x_3) \\ &> 1 - g_3(x_1', x_2', x_3) = G_1(x_1', x_2') + G_1(x_1', x_2'), \end{aligned} \quad (5)$$

where we used again the monotonicity property (B) for $g_3$ and (separately) for the variables $x_1 \geq x_1'$ and $x_2 \geq x_2'$ (note that here we cannot have equality in both places, therefore, the strict inequality holds in the second step of (5)). This again shows that $(G_1(x_1, x_2), G_1(x_1, x_2)) \neq (G_1(x_1', x_2'), G_1(x_1', x_2'))$, and hence we have established that the mapping (4) is a homeomorphism.

Analogous formulation and proofs hold for the rational decision problem when there are $n$ brothers.

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