

Cubic Surfaces and Linear Recurrences

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1 Introduction

Ramanujan [11, p. 341] found solutions of the Diophantine equation

$$x^3 + y^3 = z^3 + t^3 \quad (1)$$

for which $t = \pm 1$ and x, y, z are the n th terms in certain sequences. These sequences form linearly independent solutions of the linear recurrence with companion polynomial $(s+1)(s^2 - 83s + 1)$. (See [8, p. 63]).

Hirschhorn [5], [7] derives the result from a preliminary solution of (1) in binary quadratic forms, and suggests that Ramanujan's approach was the same. The present paper identifies the geometrical locus of these solutions then recalls some closely allied work, mentioned in Dickson's History but uncommon in Australian libraries.

2 Quadratic forms

In [10, vol. 2, p. 266] Ramanujan observes (albeit more concisely) that

$$n[(mp + nq)^3 - (nq + mr)^3]$$

or, expanded,

$$mn(p-r)[m^2(p^2 + pr + r^2) + 3mn(p+r)q + 3n^2q^2],$$

is a symmetric function of m and n when (by inspection)

$$p^2 + pr + r^2 = 3q^2,$$

a condition independent of m and n . To solve this equation write

$$pw - r\omega^2 = (\omega - \omega^2)(a\omega + b\omega^2)^2,$$

where $\omega^2 + \omega + 1 = 0$, then equate coefficients of ω and ω^2 . Expressed symmetrically, the result is

$$p + 3a^2 = q + 3ab = r + 3b^2 = (a + b)^2.$$

To provide a solution of equation (1), m and n must be proportional to cubes. But the equation reduces to an identity when they are ± 1 . The next simplest choices include $m = 8$, $n = 1$. After removing a common factor 3, we have then

$$\begin{aligned} x &= (3, 5, -5), y = (4, -4, 6), \\ z &= (-5, 5, 3), t = (6, -4, 4), \end{aligned}$$

(λ, μ, ν) serving as an abbreviation for the binary quadratic form $\lambda a^2 + \mu ab + \nu b^2$.

The same identity reappears in "Question 441" [9, p. 326]. See also "Question 661" and "Question 681" (p. 331), the "Comments on the papers" (pp. 393, 399, 400), and [10, vol. 2, p. 4]. Hirschhorn uses this solution of equation (1) with $a + b, a - 2b$ replacing a, b . So his claim [5, p. 200] to reconstruct the path that led Ramanujan to equation (1) seems beyond doubt.

However, binary quadratic forms span a vector space of dimension 3, whence x, y, z, t above must be linearly dependent. In fact, by a mental calculation,

$$x - z = 4(t - y). \tag{2}$$

Consequently, all solutions in the parametric form above are confined to the plane given by equation (2). This plane meets the cubic surface (1) in the line $x = z, t = y$, and residually in the conic where the plane intersects the quadric surface

$$4(x^2 + xz + z^2) = t^2 + ty + y^2. \tag{3}$$

Now, a straightforward calculation shows that x, y, z, t above satisfy (3) anyway, whence the solutions of (1) in this parametric form are restricted entirely to a conic. It meets the straight line in just two points, given by $a = \pm b$.

3 Linear recurrences

In paraphrase, the recurrent solutions of equation (1) arise from specialising the values a, b in a solution by quadratic forms, while these forms satisfy lower-degree equations having equation (1) as just one consequence among many. Now, the statement in [11, p. 341] (quoted in [5, p. 199]) excludes reference to quadratic forms, without whose aid the solution itself may be hard to derive. But can we anticipate directly from the form of the original assertion at least its dependence on equations of lesser degree? I show next how it follows from the reducibility of the companion polynomial.

Thus, let $f(s) = (1 - \alpha s)(1 - \beta s)(1 - \gamma s) = 1 - As + Bs^2 - Cs^3$ with s an indeterminate, α, β, γ distinct and A, B, C rational. Write p_0, p_1, p_2, \dots for the coefficients in the Taylor expansion

$$1/f(s) = \sum p_n s^n. \tag{4}$$

Multiplying both sides by $f(s)$ and equating coefficients of like powers of s gives

$$p_n - Ap_{n-1} + Bp_{n-2} - Cp_{n-3} = 0, n \geq 3, \tag{5}$$

whence the coefficients satisfy a linear recurrence relation with constant coefficients. Instead, multiply both sides by $(1 - \beta s)(1 - \gamma s)$ and equate coefficients of like powers of s in the expanded forms to obtain

$$p_n - (\beta + \gamma)p_{n-1} + \beta\gamma p_{n-2} = \alpha^n \tag{6}$$

and, by symmetry,

$$p_n - (\gamma + \alpha)p_{n-1} + \gamma\alpha p_{n-2} = \beta^n, \tag{7}$$

$$p_n - (\alpha + \beta)p_{n-1} + \alpha\beta p_{n-2} = \gamma^n. \tag{8}$$

For example, if $\alpha = 0, A = 9$ and $B = -1$ then equations (5)–(6) both reduce to

$$p_n - 9p_{n-1} - p_{n-2} = 0,$$

while multiplying corresponding sides of equations (7)–(8) produces

$$p_n^2 - 9p_n p_{n-1} - p_{n-1}^2 = (-1)^n.$$

Several results for the case $C = \alpha\beta\gamma = 1$ also follow easily. Suppose first that $f(s)$ is irreducible over the rational numbers. Multiply corresponding sides of equations (6)–(8) to produce

$$\prod [\rho p_n - \rho(A - \rho)p_{n-1} + p_{n-2}] = 1, \tag{9}$$

the three factors being for $\rho = \alpha, \beta, \gamma$. Hence (p_n, p_{n-1}, p_{n-2}) is a point on a cubic surface (now in Cartesian coordinates) defined over the rational numbers.

If, instead, $f(s)$ is the product of the linear factor $(1 - s)$ with an irreducible quadratic then we may take $\alpha = \beta\gamma = 1$. In place of equation (9) there are two equations with rational coefficients, both formally expressible as (9), but with either a single factor, for $\rho = \alpha$, or with two factors, for $\rho = \beta, \gamma$. Thus, if $f(s) = (1 - s)(1 + 83s + s^2)$ we find

$$p_n + 83p_{n-1} + p_{n-2} = 1, \tag{10}$$

$$(p_n - p_{n-1})^2 + 83(p_n - p_{n-1})(p_{n-1} - p_{n-2}) + (p_{n-1} - p_{n-2})^2 = 1. \tag{11}$$

The point with coordinates (p_n, p_{n-1}, p_{n-2}) lies on the conic determined by the common solutions of equations (10), (11). No third relation among the three coordinates can be algebraically independent of these two, because three independent equations would have only finitely many solutions.

Equation (4) implies that

$$(\lambda + \mu s + \nu s^2)/f(s) = \sum A_n s^n, \quad A_n = \lambda p_n + \mu p_{n-1} + \nu p_{n-2}.$$

All such sequences A_n solve the one linear recurrence (5). Moreover, in a relation such as (6) or (9) we can eliminate (p_n, p_{n-1}, p_{n-2}) in favour of (A_n, B_n, C_n) , where the components of this vector come from three linearly independent triples (λ, μ, ν) . Conversely, the coordinates of any such points (A_n, B_n, C_n) can be expressed linearly in terms of p_n, p_{n-1}, p_{n-2} , whence all these points lie on a conic.

This conclusion is not peculiar to solutions of equation (1), but valid for all sets of three independent solutions of a cubic recurrence with reducible companion polynomial. Nevertheless, in the Appendix I carry out the calculations for the example above, then apply them to deduce Ramanujan’s result explicitly from equations of lesser degree.

Thus we see again (here from reducibility, as above from a dimension argument involving quadratic forms) that Ramanujan’s identity (1) can be only one member, by no means of least degree, in a family of identities satisfied by the n^{th} terms of the same sequences. The relation of least degree for the sequences in [11, p. 341], namely $a_n + 4b_n - 4c_n = (-1)^n$, can be verified numerically for the first seven triples, given in [6]. This relation follows from the identity

$$(1 + 53s + 9s^2) + 4(2 - 26s - 12s^2) - 4(2 + 8s - 10s^2) = 1 - 83s + s^2,$$

on dividing both sides by $(1 + s)$ times the right member. Cf. equation (2).

4 A cognate identity

As a second application of the theory developed above, I note a further result concerning recurrence-derived points on a cubic surface. Thus, subtracting (8) from (7) yields

$$(\beta - \gamma)(p_{n-1} - \alpha p_{n-2}) = \beta^n - \gamma^n,$$

two more relations following by symmetry. Hence by multiplying corresponding sides,

$$p_{n-1}^3 f(p_{n-2}/p_{n-1}) = R(\alpha, \beta, \gamma)$$

where [1, p. 111ff.] the right side is the bialternant $|\alpha^0 \beta^n \gamma^{2n}|/|\alpha^0 \beta^1 \gamma^2|$. If now we write

$$g(s) = (1 - s/\alpha)(1 - s/\beta)(1 - s/\gamma), \quad 1/g(s) = \sum q_n s^n$$

then, by the same reasoning, $q_{n-1}^3 g(q_{n-2}/q_{n-1})$ will be

$$R(1/\alpha, 1/\beta, 1/\gamma) = (\alpha\beta\gamma)^{2-2n} R(\alpha, \beta, \gamma).$$

Moreover, we have $(\alpha\beta\gamma)g(s) = -s^3f(1/s)$. On eliminating R and replacing n by $n + 1$ we get

$$p_n^3 f(p_{n-1}/p_n) + C^{2n-1} q_{n-1}^3 f(q_n/q_{n-1}) = 0.$$

For a self-reciprocal cubic $f(s) = s^3f(1/s)$ the result is vacuous. But if $f(s) = 2 - (1+s)^3$ we see that $(x, y, z, t) = (a_n, b_n, c_n, d_n) = \mathbf{a}^T$ is a point of the surface

$$x^3 + y^3 = 2(z^3 + t^3), \quad (12)$$

a_n, b_n, c_n, d_n being the respective coefficients of s^n in the power-series expansions of the functions

$$(1+s)/f(s), (1+s)/g(s), 1/f(s), s/g(s).$$

This result, chronicled in [4, p. 562], is proved in [2, p. 282–285]. Might Ramanujan have seen it? The first four solution vectors \mathbf{a}^T are $(1, 1, 1, 0)$, $(4, -2, 3, 1)$, $(15, 3, 12, -3)$ and $(58, -2, 46, 6)$. They form the rows of a matrix with determinant 36. So the points \mathbf{a} are non-coplanar. Again, as the first 10 matrices $\mathbf{a}\mathbf{a}^T$ are linearly independent, these points do not all lie on a quadric surface. Both a_9 and c_9 have 6 digits, the relevant determinant being a number with 30 digits. I have not attempted to ascertain whether the first 20 points might lie also on a second cubic surface besides (12).

In [3] I used a parametric solution of equation (12), though differently derived, to construct quadratic number fields with unusual ideal class groups.

Appendix

$$\begin{aligned} A_n &= p_n - 53p_{n-1} + 9p_{n-2} \\ B_n &= 2p_n + 26p_{n-1} - 12p_{n-2} \\ C_n &= 2p_n - 8p_{n-1} - 10p_{n-2} \end{aligned}$$

and let U_n, V_n denote the left sides of equations (10), (11). Then equation (1) has the solution $(x, y, z, t) = (A_n, B_n, C_n, 1)$.

In fact, $B_n - C_n = 2(17p_{n-1} - p_{n-2})$, $U_n - A_n = 8(17p_{n-1} - p_{n-2}) = 4(B_n - C_n)$ (cf. equation (2)). To verify $B_n^3 - C_n^3 = U_n^3 - A_n^3$ it is then sufficient to check that $B_n^2 + B_nC_n + C_n^2$, or $(B_n - C_n)^2 + 3B_nC_n$, is four times $(U_n - A_n)^2 + 3U_nA_n$. This condition reduces to $p_n p_{n-1} + p_{n-1} p_{n-2} + p_{n-2} p_n + 82p_{n-1}^2 = 0$. By a similar reduction, so too does the equation $U_n^2 = V_n$, whence the truth of both equations is immediate from (10), (11).

Ramanujan's result follows on writing $a_n = (-1)^n A_n$ etc. Varying the sign (use of $f(s)$ instead of $f(-s)$) allowed us to express all our solutions of (1) with $t = 1$ and hence interpret (x, y, z) as Cartesian coordinates. Ramanujan avoids this simplification, presumably to secure sequences of positive terms.

Finally, I reiterate that the purpose here is not to offer an alternative derivation but merely to confirm that, among many similar relations, the result follows from simpler, more fundamental equations.

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