Binomial formula and polynomial approximations

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Abstract

We use the binomial formula to obtain approximations to some often used functions and our approach yields an elementary and fairly simple proof of the Weierstrass theorem on polynomial approximations of continuous functions.

1 Half the binomial formula

Using the binomial formula we obtain

$$2^m = \sum_{k=0}^{m} \binom{m}{k} (1+x)^k (1-x)^{m-k}.$$

For an odd m, say $m = 2n + 1$, we take the first $n + 1$ terms of this expansion for a definiton of interesting functions B_{2n+1}. We consider odd m for reasons of symmetry, the binomial formula contains an even number of terms and we wish to retain exactly half of them.

$$B_{2n+1}(x) = 2^{-2n-1} \sum_{k=0}^{n} \binom{2n+1}{k} (1+x)^k (1-x)^{2n+1-k}.$$

It is easy to see that $B_{2n+1}(x) + B_{2n+1}(-x) = 1$, $0 \le B_{2n+1}(x) \le 1$ for $x \in [-1, 1]$, $B_{2n+1}(-1) = 1$, $B_{2n+1}(1/2) = 1/2$ and $B_{2n+1}(1) = 0$. Graphs of some functions B_{2n+1} are in Figure 1.

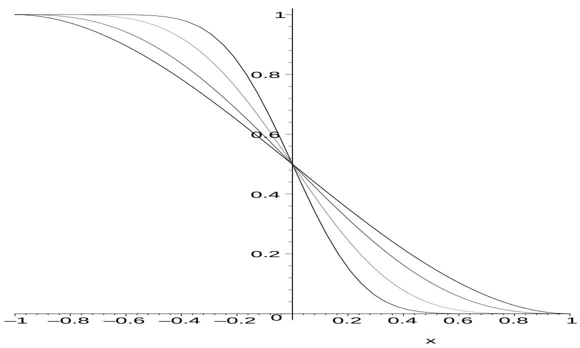

Figure 1. Graphs of B_3, B_5, B_{11}, B_{25}

These graphs suggest that $B_{2n+1}(x) \to 1$ for $-1 \le x < 0$ and $B_{2n+1}(x) \to 0$ for $0 < x \le 1$ as $n \to \infty$.There is also a compact formula for the derivative

$$B'_{2n+1}(x) = -2^{-2n-1}\binom{2n+1}{n}(n+1)(1-x^2)^n,$$

but we do not need it.

2 Convergence

For the proof of convergence mentioned at the end of the previous section we note that the powers of $1-x$ are dominant. We have

$$(1+x)^k \le (1+x)^n, \qquad (1-x)^{2n+1-k} \le (1-x)^{n+1}$$

for $0 < x \le 1$ and $k = 0, 1, \ldots, n$. Consequently

$$B_{2n+1}(x) \le (1+x)^n(1-x)^{n+1}2^{-2n-1}\sum_{k=0}^{n}\binom{2n+1}{k} \le (1-x^2)^n$$

It follows that $B_{2n+1}(x) \to 0$ uniformly on $0 < \Delta \le x \le 1$. Since $B_{2n+1}(x) = 1 - B_{2n+1}(-x)$ we also have that $B_{2n+1}(x) \to 1$ uniformly on $[-1, -\Delta]$. By replacing x with $(t-c)/(b-a)$ we have the following

Lemma 1 *If $a < c < b$ then for every positive ε and $0 < \delta < Min(c-a, b-c)$ there exists a polynomial E such that*

$$\begin{aligned} 0 \le E(t) &\le 1 && \text{for } a \le t \le b,\\ 0 \le 1 - E(t) &< \varepsilon && \text{for } a \le t \le c-\delta,\\ 0 \le E(t) &< \varepsilon && \text{for } c+\delta \le t \le b. \end{aligned}$$

This leads to

Lemma 2 *If $a = c_0 < c_1 < c_2 < \cdots < c_{n-1} < c_n = b$ and $P_1, P_2, \ldots P_n$ are polynomials then for every ε satisfying*

$$|P_2(c_1) - P_1(c_1)| + |P_3(c_2) - P_2(c_2)| + \cdots + |P_n(c_{n-1}) - P_{n-1}(c_{n-1})| < \varepsilon$$

there is a polynomial P such that

$$|P(x) - P_i(x)| < \varepsilon \qquad \text{for } x \in [c_{i-1}, c_i]$$

and $i = 1, 2 \ldots n$.

Proof We give the proof only for $n = 2$, the general case follows by induction. By continuity there exists $\delta > 0$ such that $|P_2(x) - P_1(x)| < \varepsilon$ for $c_1 - \delta < x < c_1 + \delta$. We choose a constant M larger than $|P_1| + |P_2|$ on $[a, b]$. Let E be as in Lemma1 with $c = c_1$ and ε replaced by ε/M. Define $P(x) = E(x)P_1(x) + (1 - E(x))P_2(x)$. Clearly $P(x) - P_1(x) = [1 - E(x)][P_1(x) - P_2(x)]$. If $c_0 \le x \le c_1 - \delta$ then $0 \le 1 - E(x) < \varepsilon/M$, if $c_1 - \delta < x \le c_1$ then $0 \le 1 - E(x) \le 1$ and $|P_1(x) - P_2(x)| < \varepsilon$. In either case $|P(x) - P_1(x)| < \varepsilon$ for $x \in [c_0, c_1]$. The inequality $|P(x) - P_2(x)| < \varepsilon$ on $[c_1, c_2]$ follows similarly.

It is now easy to obtain polynomial approximations to some often occuring functions like $x \mapsto |x|$ or

$$x_+(t) = \begin{cases} 0, & \text{for } t < 0\\ t, & \text{for } t \ge 0. \end{cases}$$

Graphs of these approximation are in Figure 2. Uniform approximation of $|x|$ are useful in proving many theorems in classical approximation theory.

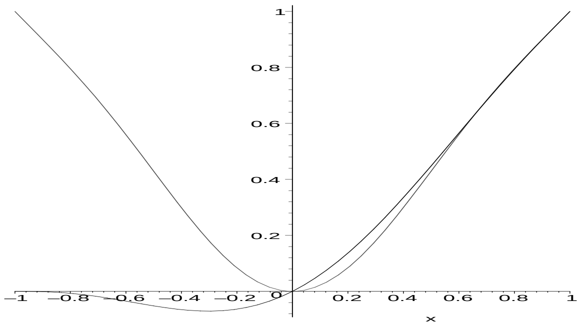

Figure 2. Graphs of $xB_5(-x)$, $(-x)B_7(x) + xB_7(x)$

3 Proof of the Weierstrass Theorem

An immediate consequence of Lemma 2 is

Theorem 1 *A function continuous and piecewise polynomial on* $[a, b]$ *can be uniformly approximated by a polynomial.*

By [1] Theorem 12.20 (or by using uniform continuity) a function continuous on a bounded closed interval can be uniformly approximated by a continuous and piecewise linear function. By the above Theorem this function can be uniformly approximated by a polynomial. Hence the function itself can be uniformly approximated by a polynomial.

Remark Our proof has some similarity with the proof using the Bernstein's polynomials, compare [2] but is perhaps more suitable for undergraduates. An elementary proof of the Weierstrass theorem are also given in [3] and [1].

References

[1] P. Adams, K. Smith and R. Výborný, *Introduction to Mathematics with Maple* (World Scientific 2004), 344–345.
[2] I.P. Natanson, *Theory of Functions of a Real Variable*, (Engl. transl.) Vol 1, (Ungar New York 1974).
[3] H. Kuhn, *Ein elementarer Beweis des Weierstrassischen Approximationsatzes*, Arch. Math. **15** (1964), 316–317.

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