On representations of π and $\sqrt{2}$

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In [1] A. Sofo has given us many formulas for such numbers as π, $1/\pi$, π^2, π^4, $\pi/\sqrt{3}$, $\pi/(\sqrt{6}-\sqrt{2})$ and $\sqrt{640320}/\pi$ in terms of infinite sums of rationals. The questions I would like to discuss are, which is the "best" for computational purposes, and, can we do better?

Suppose that $\alpha \in \mathbb{R}$, and that

$$\alpha = \sum_{k=1}^{\infty} a_k,$$

where a_k is rational.

Let $x_n = \sum_{k=1}^{n} a_k$. Then $x_n \to \alpha$ as $n \to \infty$. In order to discuss the rate of convergence, let us define the discrepancy δ_n by

$$\delta_n = |\alpha - x_n|.$$

We prove the following

Theorem 1 *If $a_{n+1}/a_n \to r$ as $n \to \infty$ where $|r| < 1$, then $\delta_{n+1}/\delta_n \to |r|$ as $n \to \infty$.*

Proof. We have

$$\begin{aligned}
\delta_{n+1}/\delta_n &= \left|\alpha - \sum_{k=1}^{n+1} a_k\right| \Big/ \left|\alpha - \sum_{k=1}^{n} a_k\right| \\
&= \left|\sum_{k=n+2}^{\infty} a_k\right| \Big/ \left|\sum_{k=n+1}^{\infty} a_k\right| \\
&= \frac{|a_{n+2}|}{|a_{n+1}|} \left| \frac{1 + \dfrac{a_{n+3}}{a_{n+2}} + \dfrac{a_{n+3}}{a_{n+2}}\dfrac{a_{n+4}}{a_{n+3}} + \cdots}{1 + \dfrac{a_{n+1}}{a_n} + \dfrac{a_{n+1}}{a_n}\dfrac{a_{n+2}}{a_{n+1}} + \cdots} \right| \\
&\to |r| \left| \frac{1 + r + r^2 + \cdots}{1 + r + r^2 + \cdots} \right| \\
&= |r|.
\end{aligned}$$

□

Most of the sum–type examples given by Sofo are of the above type, for various values of r. Obviously, the smaller $|r|$, the greater the rate of convergence. Indeed, every T terms of the series will give an extra $T\log(1/|r|)$ decimal places of accuracy in α. Thus, in several of his examples, $r = 1/4$, so every 5 terms will give 3 decimal places of accuracy.

By far the fastest converging series he gives is one for calculating $\sqrt{640320}/\pi$ due to the Chudnovskys, in which $r = 1/53360^3$, and every term gives 14 extra decimal places.

But note that no matter the value of r, if $|r| > 0$, the number of extra decimal places given by each additional term is (roughly) constant. We can sometimes do better than that! Let me give a couple of examples.

It can be shown that, if the sequence of integers $\{a_k\}$ is given by

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 21, \quad a_{n+3} = 5a_{n+2} + 5a_{n+1} - a_n$$

then

$$\sqrt{2} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{a_k},$$

and if we define $x_n = 2\sum_{k=1}^{n}(-1)^{k-1}/a_k$, $\delta_n = |\sqrt{2} - x_n|$, then

$$\delta_{n+1}/\delta_n \to \frac{1}{3+2\sqrt{2}} \quad \text{as} \quad n \to \infty,$$

so every 4 terms give 3 extra decimal places.

On the other hand, if we define the sequence of integers $\{a_k\}$ by

$$a_1 = 2, \quad a_2 = 12, \quad a_{k+2} = 2a_{k+1}(4a_k^2 + 1),$$

then

$$\sqrt{2} = 2 - \sum_{k=1}^{\infty} \frac{1}{a_k},$$

and if we define $x_n = 2 - \sum_{k=1}^{n} 1/a_k$,

$$\delta_{n+1} < \frac{1}{2\sqrt{2}}\,\delta_n^2.$$

Such convergence is said to be quadratic, and means that with each extra term, the number of decimal places of accuracy (at least) doubles! Thus

$$x_1 = 2 - \frac{1}{2} = 1.5,$$
$$x_2 = 2 - \frac{1}{2} - \frac{1}{12} = \mathbf{1.41}66...\ ,$$
$$x_3 = \mathbf{1.41421}568...\ ,$$
$$x_4 = \mathbf{1.41421356237}468...\ ,$$
$$x_5 = \mathbf{1.414213562373095048801688}962...\ ,$$
$$x_6 = \mathbf{1.4142135623730950488016887242096980785696718753 7}732...\ ,$$

and so on.

As a third example, if we define the integer sequences $\{a_k\}$, $\{b_k\}$ by

$$a_1 = b_1 = 1, \quad a_{k+1} = 4a_k^3 + 3a_k, \quad b_{k+1} = 8b_k^3 - 3b_k,$$

then

$$\sqrt{2} = 1 + 2\sum_{k=1}^{\infty} \frac{a_k}{b_{k+1}},$$

and if $x_n = 1 + 2\sum_{k=1}^{n} a_k/b_{k+1}$ then

$$\delta_{n+1} \sim \frac{1}{8}\,\delta_n^3,$$

the convergence is cubic, and the number of decimal places of accuracy more than triples with each term!

$$\begin{aligned} x_1 &= \mathbf{1.4}, \\ x_2 &= \mathbf{1.414213}197... \ , \\ x_3 &= \mathbf{1.414213562373095048}795... \ , \\ x_4 &= \mathbf{1.414213562373095048801688724209 6} \\ &\qquad \mathbf{980785696718753769480731766797}103... \ . \end{aligned}$$

One can, alternatively, give iterative processes for $\sqrt{2}$, which converge to any given order. For instance, here is one that converges to the 10th order, that is, each iteration gives 10 times as many decimal places of accuracy.
Let

$$x_{n+1} = \frac{x_n^{10} + 90x_n^8 + 840x_n^6 + 1680x_n^4 + 720x_n^2 + 32}{10x_n^9 + 240x_n^7 + 1008x_n^5 + 960x_n^3 + 160x_n},$$

and begin with x_1 near $\sqrt{2}$, say $x_1 = 1.41421$. Then

$$x_2 = \mathbf{1.4142135623730950488016887242096980785696718753769480731766797}08...$$

and x_3 agrees with $\sqrt{2}$ to more than 500 decimal places! I have found iterative processes for π of the third and 5 orders, and so on. Their implementation requires some detailed explanation. Let me give one of the third order:

$$x_{n+1} = x_n + 2\cos\left(\frac{x_n}{2}\right).$$

To reduce the argument of the cosine term to less than 1 (I assume we will begin the iteration near π), write

$$x_{n+1} = x_n + 2(2(2(2(2\cos^2\left(\frac{x_n}{32}\right) - 1)^2 - 1)^2 - 1)^2 - 1).$$

Next, if you want $N(> 10)$ decimals, replace the cosine term by the polynomial

$$\sum_{k=0}^{M} \frac{(-1)^k}{(2k)!}\left(\frac{x_k}{32}\right)^{2k}.$$

where $\dfrac{1}{(2M)!} < \dfrac{1}{10^N}$.
So

$$x_{n+1} = x_n + 2(2(2(2\left(\sum_{k=0}^{M} \frac{(-1)^k}{(2k)!}\left(\frac{x_k}{32}\right)^{2k}\right)^2 - 1)^2 - 1)^2 - 1)^2 - 1).$$

With $x_1 = 3.14$, $N = 10$ and $M = 7$, this gives

$$x_2 = \mathbf{3.1415926}41... \ ,$$

with $x_2 = 3.141593$, $N = 20$ and $M = 11$,

$$x_3 = \mathbf{3.14159265358979323 8}9... \ ,$$

and so on, in a matter of seconds.

References

[1] A. Sofo, *Some representations of* π, Aust. Math. Soc. Gazette **31** (2004), 184–189.

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