The 9th Problem

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In the year 2000, exactly one hundred years after David Hilbert posed his now famous list of 23 open problems, The Clay Mathematics Institute (CMI) announced its seven Millennium Problems. (http://www.claymath.org/millennium). Any person to first publish a correct solution, proof or disproof of one of the following problems: 1) Birch and Swinnerton–Dyer Conjecture, 2) Hodge Conjecture 3) Navier–Stokes Equations 4) P versus NP 5) Poincaré Conjecture 6) Riemann Hypothesis 7) Yang–Mills Theory, does not only earn immortal fame but will be awarded the generous sum of one million US dollars. With Perelman's (likely) proof of the Poincaré Conjecture, the continued optimism about an impending proof of the Riemann Hypothesis, and the omission of such famous problems as Twin Primes and Goldbach, it seems the CMI would have been wise to have followed Hilbert's example in announcing not 7 but 23 Millennium Problems. The Gazette will try to repair the situation, and has asked leading Australian mathematicians to put forth their own favourite 'Millennium Problem'. Due to the Gazette's limited budget, we are unfortunately not in a position to back these up with seven-figure prize monies, and have decided on the more modest 10 Australian dollars instead.
In this issue John Urbas will explain his favourite open problem that should have made it to the list.

Nonlinear elliptic equations

In this article I will describe a longstanding open problem whose resolution will be a major development in the theory of nonlinear second order elliptic equations, and which I think is of sufficient importance and depth to be included in the *Gazette's* Millennium Series. The question can be phrased simply as follows. Under what conditions (beyond natural ellipticity and regularity assumptions to be described below) are solutions u of an elliptic equation

$$F(D^2u) = g(x) \tag{1}$$

(interpreted in a suitable generalized sense) necessarily classical solutions, that is, solutions having continuously differentiable second derivatives? This is known to be true in two dimensions with no structure condition on F because of certain estimates proved in the 1950s, while in higher dimensions it was proved in the 1980s under the assumption that F is concave. We would like to know just how much the concavity condition can be weakened, or whether it can be eliminated altogether. A resolution of this problem will either lead to the existence of classical solutions for various important equations for which this is not known at present, or else, identify a fundamental structural obstruction to the existence of classical solutions.

To discuss the problem further we need to explain the terms used above. In (1) F is a smooth function defined on the space of symmetric $n \times n$ matrices $\mathcal{S}^{n\times n}$, or on some open

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subset of $\mathcal{S}^{n\times n}$, and g is a smooth function defined on some bounded open set $\Omega \subset \mathbf{R}^n$, usually having a smooth boundary $\partial\Omega$. A *classical solution* u of (1) is a twice continuously differentiable function $u : \Omega \to \mathbf{R}$ (we write this as $u \in C^2(\Omega)$) which satisfies equation (1) at each point of Ω. In (1) D^2u denotes the Hessian or matrix of second partial derivatives of u. We say that (1) is *elliptic* with respect to a given solution u if $[F^{ij}] := [\partial F(D^2+u)/\partial u_{ij}]$ is a positive definite matrix at each point of Ω. This means that the differential operator $\sum_{i,j=1}^n F^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ is second order in every direction. We say that (1) is *uniformly elliptic* with respect to a given solution u if there are positive constants $\Lambda \geq \lambda$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n F^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all} \quad \xi \in \mathbf{R}^n$$

at each point of Ω.

The best known example is the linear equation

$$\Delta u = g(x), \tag{2}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. It is obviously uniformly elliptic on any solution, with $[F^{ij}]$ equal to the identity matrix.

Another important example is the Monge-Ampère equation

$$\det D^2u = g(x). \tag{3}$$

In this case $[F^{ij}]$ is the matrix of cofactors of D^2u, so (3) is elliptic on functions u with the property that all the eigenvalues of D^2u have the same sign. By replacing u by $-u$ if necessary we may assume that D^2u is positive definite. For such solutions to exist we must take g positive. Equation (3) is uniformly elliptic on solutions u such that the eigenvalues of D^2u are bounded between two positive constants. In particular, any solution $u \in C^2(\Omega)$ with $D^2u > 0$ is uniformly elliptic on any compact subset of Ω.

If we denote the eigenvalues of D^2u by $\lambda_1, \ldots, \lambda_n$, then evidently

$$\Delta u = \sum \lambda_j \quad \text{and} \quad \det D^2u = \prod \lambda_j.$$

It is clear that many other nonlinear equations of the form (1) can be generated by taking other symmetric functions of $\lambda_1, \ldots, \lambda_n$. Some interesting examples are obtained by considering the k-th elementary symmetric functions

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k},$$

which give rise to the k-Hessian equations

$$F_k(D^2u) = g(x),$$

of which (2) and (3) are special cases. The k-Hessian equations are elliptic on solutions $u \in C^2(\Omega)$ satisfying a condition called k-convexity; it is equivalent to $D^2u > 0$ if $k = n$, and is weaker than this if $k < n$.

Equations of the above type are usually studied in conjunction with a boundary condition. The simplest and most studied boundary condition in the theory of elliptic equations is the Dirichlet condition: we are required to find a solution u of (1) defined in a given bounded domain $\Omega \subset \mathbf{R}^n$ with $u = \phi$ on $\partial\Omega$ for a given smooth function ϕ.

The basic technique to solve the Dirichlet problem is to embed the given problem

$$F(D^2u) = g(x) \quad \text{in} \quad \Omega, \qquad u = \phi \quad \text{on} \quad \partial\Omega \tag{4}$$

into a family of problems

$$F_t(D^2u) = g_t(x) \quad \text{in} \quad \Omega, \qquad u = \phi_t \quad \text{on} \quad \partial\Omega \tag{5_t}$$

for $t \in [0,1]$, in such a way that $t = 1$ corresponds to the problem we want to solve, while $t = 0$ reduces to a problem we know to be solvable in a suitable class of functions. It turns out that an appropriate class of functions to work in is the Hölder space $C^{2,\alpha}(\overline{\Omega})$, where $\alpha \in (0,1)$. The Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ are defined as follows: Given a nonnegative integer k and $\alpha \in (0,1)$, a function f belongs $C^{k,\alpha}(\overline{\Omega})$ if and only if

$$\|f\|_{k,\alpha;\Omega} := \sum_{j=0}^{k} \sup_{x\in\Omega} |D^j f(x)| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha} < \infty.$$

We then need to show that the set $\mathcal{T}$ of $t \in [0,1]$ for which (5_t) is solvable in $C^{2,\alpha}(\overline{\Omega})$ is both open and closed. Since $\mathcal{T}$ is not empty (because $0 \in \mathcal{T}$), by connectedness of $[0,1]$ we conclude that $\mathcal{T} = [0,1]$, so our original problem (4) is solvable in $C^{2,\alpha}(\overline{\Omega})$.[1]

Showing that $\mathcal{T}$ is open uses the implicit function theorem for mapping between Banach spaces and linear elliptic theory (see [6], Chapter 17). Proving that $\mathcal{T}$ is closed depends on establishing a priori estimates for solutions u_t of (5_t) in $C^{2,\alpha}(\overline{\Omega})$. This is usually the most difficult part of the whole procedure.

Once we have proved the existence of solutions in $C^{2,\alpha}(\overline{\Omega})$, it is usually possible to get further results by approximation. For example, for less regular boundary data we can often prove the existence of solutions in $C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ [2], provided we can prove precisely these bounds for solutions of a suitable family of approximating problems.

Sometimes we are not able to prove the existence of solutions in $C^{2,\alpha}(\overline{\Omega})$ or in $C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ because we lack the appropriate estimates. In such cases the best we can usually do is prove the existence of some kind of generalized solution u and show it has some low level of regularity, for example $u \in C^{0,\alpha}(\Omega)$ or $u \in C^{1,\alpha}(\Omega)$. Again, all this depends on proving appropriate estimates.

At the begin of the article the problem was phrased as a question about the smoothness of generalized solutions. However, the key issue is really what estimates can be proved for smooth solutions. Once we have such estimates, the regularity of generalized solutions can be addressed by suitable approximation techniques. For the purposes of this article it is not necessary to know what a generalized solution is.[3]

It should be clear from the preceding discussion that estimates play a central role in the theory of partial differential equations. For this reason major bursts of progress in the subject have come about with the discovery of new estimates and new ¿¡DEFANGED.4 techniques for proving estimates. Very often these are estimates for solutions of linear equations rather than nonlinear ones, but it turns out that the linear estimates provide fundamental tools for studying nonlinear equations.

There have been several such major breakthroughs during the last fifty years. In the late 1950s De Giorgi [4], Nash [10] and Moser [9] proved a local estimate in $C^{0,\alpha}(\Omega)$ for weak

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[1]For some equations such as the Monge-Ampère equation this needs to be modified a little; we sometimes need to work in a suitable open subset of $C^{2,\alpha}(\overline{\Omega})$ rather than in the whole space. Also, the family of problems (5_t) needs to be chosen so that various structure and regularity hypotheses are satisfied uniformly for the whole family of problems.

[2]$u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ means that $u \in C^{2,\alpha}(\Omega')$ for each bounded open set $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \Omega$, and u extends continuously to $\overline{\Omega}$.

[3]The appropriate notion is that of *viscosity solution*. An exposition of this theory is given in [3].

solutions of linear, uniformly elliptic equations of divergence form

$$\sum_{i,j=1}^{n} D_i \left(a^{ij}(x) D_j u\right) = f(x)$$

with bounded measurable coefficients a^{ij}. This development rapidly opened up the theory of quasilinear equations of the form

$$Q[u] = \sum_{i,j=1} a^{ij}(x, u, Du) D_{ij} u = b(x, u, Du)$$

in dimensions $n \geq 3$. Previously the theory of such equations had been almost entirely restricted to two dimensions, where complex analytic techniques were available.

A second major development occurred in 1980. This was the derivation by Krylov and Safonov [8] of an analogue of the De Giorgi-Nash-Moser estimate for solutions of linear uniformly elliptic equations of nondivergence form

$$\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u = f(x)$$

with measurable coefficients a^{ij}.

Estimates for solutions of linear equations are useful for studying nonlinear equations because we can apply the estimates for linear equations to the first and second derivatives of a solution u of a nonlinear equation to obtain Hölder continuity estimates for the first and second derivatives of u. The equations for the derivatives of u are obtained by differentiating the original equation, so the first and second derivatives of u may not satisfy a nice equation, especially when we consider more general equations of the form $F(x, u, Du, D^2u) = 0$. Usually all we can derive is a reasonable differential inequality, so a good deal of further work is necessary.

To illustrate this we return to equation (1) and differentiate it twice in a direction ξ, obtaining

$$\sum_{i,j} F^{ij} D_{ij\xi\xi} u + \sum_{i,j,k,l} F^{ij,kl} D_{ij\xi} u D_{kl\xi} u = D_{\xi\xi} g,$$

where

$$F^{ij,kl} = \frac{\partial^2 F(D^2 u)}{\partial u_{ij} \partial u_{kl}}.$$

Notice that in general this does not give us a good differential equation for $v = D_{\xi\xi} u$ because the quadratic term in third derivatives of u cannot be expressed in terms of first derivatives of v, and certainly not in a linear way. However, if $F(D^2u)$ is a concave function of D^2u, this quadratic term is nonpositive, and we see that v satisfies the linear looking differential inequality

$$\sum_{i,j} F^{ij} D_{ij} v \geq D_{\xi\xi} g.$$

Of course, a differential inequality is much weaker than a differential equation, so it is far from obvious that anything useful can be obtained from this. Remarkably, in the 1980s Evans [5] and Krylov [7] independently used the Krylov-Safonov theory and other tools to prove a local second derivative Hölder estimate for solutions of (1) under the above concavity assumption. A more precise statement is the following: If $u \in C^2(\Omega)$ is an elliptic solution of (1) and $D^2u \mapsto F(D^2u)$ is concave, then for any concentric balls $B_r \subset B_{2r} \subset \Omega$ we have an estimate

$$[D^2u]_{\alpha;B_r} \leq \frac{C}{r^\alpha}\Big\{\|D^2u\|_{L^\infty(B_{2r})} + r\|Dg\|_{L^\infty(B_{2r})} + r^2\|D^2g\|_{L^\infty(B_{2r})}\Big\}$$

for some $\alpha \in (0,1)$ and $C > 0$ depending only on n and Λ/λ, where λ and Λ are the ellipticity constants, and

$$[D^2u]_{\alpha;B_r} = \sup_{\substack{x,y\in B_r\\ x\neq y}} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha}.$$

Fortunately, many important equations satisfy the concavity condition, or can be rewritten in a such a way that it is satisfied. For example, for the Monge-Ampère equation (3), both $(\det D^2u)^{1/n}$ and $\log\det D^2u$ are concave functions of D^2u if $D^2u > 0$. More generally, $F_k(D^2u)^{1/k}$ is a concave function of D^2u if u is k-convex. Concave equations of the form (1) can also be written as Bellman equations

$$\inf_{\alpha\in\mathcal{A}}\left\{\sum_{i,j=1}^n a_\alpha^{ij} D_{ij}u + f_\alpha(x)\right\} = 0$$

for a suitable family of symmetric positive definite matrices $[a_\alpha^{ij}]$. Bellman equations arise naturally in stochastic control theory and have been much studied from this point of view.

However, there are important examples of equations for which the concavity condition is not satisfied, and, therefore, for which the existence of classical solutions is not presently known. The first such example is Isaacs equation

$$F(D^2u) = \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left\{\sum_{i,j=1}^n a_{\alpha\beta}^{ij}(x) D_{ij}u + f_{\alpha\beta}(x)\right\} = 0, \tag{6}$$

with $a_{\alpha\beta}^{ij}$ positive definite for each α and β. This arises in stochastic games theory.

The second example is the equation

$$\sum_{i=1}^n \arctan\lambda_i = c \tag{7}$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of D^2u and c is a constant. This equation arises in the theory of special Lagrangian submanifolds in the following way. Let M be the graph in $\mathbf{C}^n \cong \mathbf{R}^n \times \mathbf{R}^n$ of a smooth map $f : \Omega \to \mathbf{R}^n$, where Ω is an open set in $\mathbf{R}^n$. M is a *Lagrangian submanifold* of $\mathbf{C}^n$ if and only if the matrix $\left[\frac{\partial f^i}{\partial x^j}\right]$ is symmetric. In particular, if Ω is simply connected, then there exists a function $u : \Omega \to \mathbf{R}$ with $Du = f$. A Lagrangian submanifold of $\mathbf{C}^n$ is called *special* if it is also a minimal submanifold. In the above situation the graph of Du is a special Lagrangian submanifold of $\mathbf{C}^n$ if and only if

$$\operatorname{Im}\big(\det(I + \sqrt{-1}\,D^2u)\big) = \text{constant},$$

where I is the identity matrix and Im denotes the imaginary part. This can be rewritten in the form (7).

Equation (7) takes different forms depending on the dimension n and the value of c. For $n = 2$ and $c = 0$ it reduces to $\Delta u = 0$, while for $n = 2$ and $c = \pi/2$ it reduces to $\det D^2u = 1$. In both these cases one has all the estimates one needs.

For $n = 3$, the c-level set $\Sigma_c = \{M \in \mathcal{S}^{n\times n} : F(M) = c\}$ of

$$F(D^2u) = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3$$

is convex if $|c| \geq \pi/2$, but it is not convex if $-\pi/2 < c < \pi/2$. So in this range the concavity condition fails. Yuan [11] was nevertheless able to establish a local second derivative Hölder estimate for solutions of (7) in three dimensions.

In higher dimensions some progress on second derivative Hölder estimates for elliptic equations of the form $F(D^2u) = 0$ has been made by Caffarelli and Yuan [2] under weaker geometric conditions than concavity. However, their condition is far from general, and the theorem of Yuan [11] is not included as a special case.

In addition, Cabré and Caffarelli [1] have proved the second derivative Hölder estimate for solutions of a class of equations including the simplest nonconvex Isaacs equation

$$\min\{L_1u, \max\{L_2u, L_3u\}\} = 0$$

where L_1, L_2, L_3 are linear elliptic constant coefficient operators of the form $L_ku = \sum_{i,j} a_k^{ij} D_{ij}u + c_k$. Their results are in fact valid for operators that can be written as the minimum of a concave operator and a convex operator of D^2u, such as

$$F(D^2u) = \min\left\{\inf_{\alpha\in\mathcal{A}} L_\alpha u, \sup_{\beta\in\mathcal{B}} L_\beta u\right\} = 0.$$

But this is still far from including the general Isaacs equation (6).

Most experts in the field seem to believe that it is not possible to dispense with the concavity condition altogether, although this is known to be possible in two dimensions because of some estimates for solutions of linear equations that are true only in two dimensions. But we are still far from understanding just how much the concavity condition can be can be weakened without losing the continuity estimate for second derivatives. For some years a counterexample to C^2 regularity for solutions of (1) in high dimensions was claimed by Nadirashvili, but his construction has not stood up to close scrutiny. A resolution of this question would be a major advance in the theory.

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