The 11th problem

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In the year 2000, exactly one hundred years after David Hilbert posed his now famous list of 23 open problems, The Clay Mathematics Institute (CMI) announced its seven Millennium Problems. (http://www.claymath.org/millennium). The Gazette has asked leading Australian mathematicians to put forth their own favourite 'Millennium Problem'. Due to the Gazette's limited budget, we are unfortunately not in a position to back these up with seven-figure prize monies, and have decided on the more modest 10 *Australian dollars instead.*
In this issue Catherine Greenhill will explain her favourite open problem that should have made it to the list.

The Cycle Double Cover Conjecture

In this note I will describe one of the most famous conjectures in graph theory, called the Cycle Double Cover conjecture (or sometimes the Double Cycle Cover conjecture). The Cycle Double Cover conjecture was independently posed by George Szekeres [13] and Paul Seymour [9]. It has connections with many areas of graph theory such as topological graph theory and graph colouring, including the Four Colour Theorem. For this reason I feel that it is worthy of inclusion in the *Gazette*'s Millenium Series. My account draws from the survey articles by Jaeger [8] and Seymour [11]. (Dan Archdeacon's webpage http://www.emba.uvm.edu/~archdeac/problems/cyclecov.htm was also useful.) For basic graph theory definitions and notation not explained here, see for example Diestel [2].

Let $G = (V, E)$ be a graph. Here V is a finite set of vertices, E is a set of undirected edges, and we do not allow loops or repeated edges. The *degree* of a vertex is the number of edges containing v, and a graph is *regular* if every vertex has the same degree. A 3-regular graph (where every vertex has degree 3) is called a *cubic* graph. A graph G is *connected* if given any two vertices v, w of G, there is a path consisting of edges of G which starts at v and ends at w. A *cycle* in G is a connected subgraph H of G which is regular of degree 2. A *cycle double cover* of G is a multiset $\mathcal{C}$ of cycles in G such that every edge in G is contained in exactly two cycles in $\mathcal{C}$ (counting multiplicities).

Unsurprisingly, the Cycle Double Cover conjecture is concerned with the existence of cycle double covers. Clearly there will be no cycle double cover in G if G has an edge e which does not belong to any cycle. Such an edge is called a *cut edge* or *bridge* of G. Deletion of a cut edge from the edge set E (without deleting any vertices) disconnects the graph into two pieces. We say that G is *bridgeless* if it does not contain a cut edge. So a graph with a cycle double cover must be bridgeless. The **Cycle Double Cover** conjecture (CDC) says that this necessary condition is also sufficient:

(CDC conjecture): Every bridgeless graph admits a cycle double cover.

(In fact Szekeres [13] stated the conjecture just for cubic graphs, while Seymour [9] gave the conjecture for the general case.)

To motivate this conjecture, consider a planar graph G. By definition, G can be drawn in the plane without any edges crossing. Now remove the graph G from the plane, leaving a set of open connected regions of space behind. These open connected regions are the *faces* of the embedding of G in the plane. Each face can be circumnavigated by taking a walk around the edges of G which border the face. The collection $\mathcal{C}$ of these walks around the faces of G gives a cycle double cover of G, *unless* one of the walks is not a cycle. There are two ways that this can happen. Firstly, an edge can be included twice in one of the walks. But this implies that G has a cut edge, so G is not bridgeless. Otherwise, a vertex may be visited more than once in some walk. But in this case, the walk can be decomposed into simple cycles. Thus the CDC conjecture is true for planar graphs.

At first glance the CDC conjecture appears easy to prove. From a graph G, form a multigraph G' by replacing each edge of G with two parallel edges. Now every vertex of G' has even degree, so by induction we can partition the edge multiset of G' into cycles. The problem is that some of these "cycles" may be of length two, and so they are not cycles of the original graph G. Similar blocks have been found in many attempts at an inductive proof.

In the search for a proof of the CDC conjecture there have been many reformulations and strengthenings proposed. We will mention a couple of them.

The *strong embedding conjecture* is a strengthening of the CDC conjecture in a topological graph theory setting. A graph G is *2-connected* if it is connected and G has no *cut vertex*; that is, a vertex v whose deletion leaves the graph disconnected. Every 2-connected planar graph can be embedded in the plane or sphere such that the walk around every face is a cycle. The strong embedding conjecture (SEC) says that:

> (Strong Embedding Conjecture): Every 2-connected graph has an embedding in some surface such that every face is bounded by a cycle.

Such an embedding provides a cycle double cover of the graph. For cubic graphs, the strong embedding conjecture and the CDC conjecture are equivalent, but for noncubic graphs the strong embedding conjecture is stronger. Various strengthenings of the strong embedding conjecture have also been considered, involving orientability and face-colourability of the embedding. For more details see [8].

A related conjecture is Tutte's *5-flow conjecture*. The theory of nowhere-zero flows was introduced by Tutte [14] to generalise theorems like the Four Colour Theorem. Assign an orientation to each edge of a graph G in an arbitrary way. (That is, each edge becomes an ordered pair of vertices.) For a vertex v of G, let $\delta^+(v)$ be the set of edges with v as the start-vertex and $\delta^-(v)$ the set of edges with v as the end-vertex. A *circulation* in G is a function $\varphi : E(G) \to \mathbb{R}$ such that for every vertex v,

$$\sum_{e\in\delta^+(v)} \varphi(e) = \sum_{e\in\delta^-(v)} \varphi(e).$$

If $k \geq 1$ is an integer, a *nowhere-zero k-flow* in G is a circulation φ such that $|\varphi(e)| \in \{1, 2, \ldots, k-1\}$. If an undirected graph G has a nowhere-zero k-flow for some orientation of its edges then it has one for every orientation (just replace $\varphi(e)$ by $-\varphi(e)$ if the direction of the edge e is changed). Thus the property of having a k-flow is really a property of undirected graphs. Note that a graph with a cut edge cannot have a nowhere-zero k-flow, for any positive integer k.

The definition of nowhere-zero k-flows is motivated by the problem of colouring regions of a planar map. If G is a planar graph, then the faces of an embedding of G in the plane can be coloured with k colours, such that no two faces which share an edge have the same colour, if and only if G has a nowhere-zero k-flow. The proof is simple: suppose first that each face F of an embedding of the planar graph G in the plane is coloured with a colour $\alpha(F) \in \{1, \ldots, k\}$ such that no two faces sharing an edge have the same colour. Give each edge e an arbitrary orientation and define $\varphi(e) = \alpha(F_1) - \alpha(F_2)$, where F_1 is the face to the left of e and F_2 is the face to the right of e (with respect to the orientation of e). This defines a nowhere-zero k-flow of G. The converse is similarly proved (see [11]). In particular, the Four Colour Theorem implies that every bridgeless planar graph has a nowhere-zero 4-flow. This is not true in general for non-planar bridgeless graphs, as the *Petersen graph* (shown in Figure 1) does not have a nowhere-zero 4-flow. (In graph theory, the Petersen graph is everybody's favourite counterexample.)

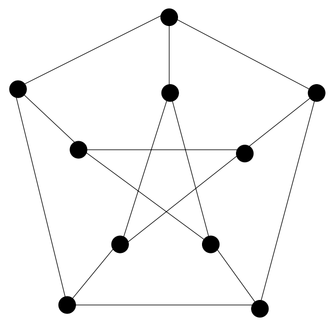

Figure 1. The Petersen graph

However, Jaeger [6] proved that every bridgeless graph has a nowhere-zero 8-flow, and this was improved when Seymour [10] showed that every bridgeless graph has a nowhere-zero 6-flow. Tutte [14] conjectured that

(Tutte 5-flow conjecture): Every bridgeless graph has a nowhere-zero 5-flow.

There are strong connections between nowhere-zero k-flows and certain types of cycle double covers. Say that a cycle double cover $\mathcal{C}$ of a graph G is *k-colourable* if we can assign a colour from the set $\{1, \dots, k\}$ to each cycle in $\mathcal{C}$ such that cycles which share an edge have distinct colours. Note that the Four Colour Theorem implies that every connected bridgeless planar graph has a 4-colourable cycle double cover. (The cycle obtained by walking around a given face is assigned the colour of that face.) Jaeger [8] proved that a graph G has a nowhere-zero 4-flow if and only if it has a 3-colourable double cover, if and only if it has a 4-colourable double cover. Similar characterisations for nowhere-zero 2-flows and nowhere-zero 3-flows are also described in [8].

Finally, let us suppose that the Cycle Double Cover is false. Then there exists a minimal counterexample, which is a bridgeless graph with no cycle double cover and with the minimal number of edges possible. A *proper edge colouring* of a graph G is formed by assigning a colour to each edge of G so that edges with a common vertex are coloured with distinct colours. Jaeger [8] proved that any minimal counterexample to the CDC conjecture is a *snark*; that is, a connected, bridgeless cubic graph which cannot be properly edge-coloured with three colours. (Snarks must also satisfy a further mild connectivity condition which we omit here.) For a long time, the Petersen graph (discovered in 1891) was the only known snark. Then two more examples, both with 18 vertices, were found by Blanuša [1] in 1946. A few more snarks were discovered (including the *Szekeres snark* [13] on 50 vertices) before Isaacs [5] produced two infinite families of snarks. (The fact that examples of these graphs were so hard to find for so long prompted Martin Gardner (in 1975) to suggest giving them the name of "snark", after the creature from the poem "The Hunting of the Snark" by Lewis Carroll.) Figure 2 shows the two Blanuša snarks and the so-called "flower" snark.

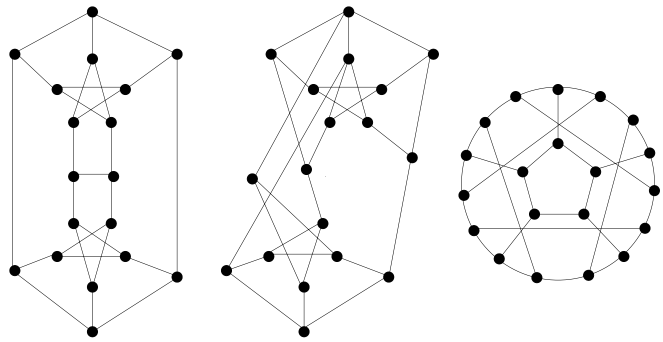

Figure 2. The two Blanuša snarks and the flower snark

More is known about minimal counterexamples to the CDC conjecture. The *girth* of a graph is the number of edges in its smallest cycle. Goddyn [3] proved that the smallest counterexample to the Cycle Double Cover conjecture must be a snark with girth at least seven. But Jaeger and Swart [7] conjectured that every snark has girth at most six, which if true would prove the CDC conjecture.

To conclude, the CDC conjecture, strong embedding conjecture and Tutte's 5-flow conjecture all motivate research today (see for example [4, 12]). In other words, the hunting of the (minimal counterexample) snark continues.

References

[1] D. Blanusa, *Problem cetiriju boja*, Glasnik Mat. Fiz. Astr. Ser. II **1** (1946), 31–42.

[2] R. Diestel, *Graph Theory* (Springer Berlin 2000). Also available at http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/download.html

[3] L. Goddyn, *A girth requirement for the double cycle cover conjecture*, in *Cycles in Graphs* (B. Alspach and C.D. Godsil, eds.), Annals of Discrete Math.**27** (North-Holland Amsterdam 1985), 13–26.

[4] R. Häggkvist and S. McGuiness, *Double covers of cubic graphs with oddness 4*, J. Combin. Theory Ser. B **93** (2005), 251–277.

[5] R. Isaacs, *Infinite families of nontrivial trivalent graphs which are not Tait colorable*, Amer. Math. Monthly **82** (1975), 221–239.

[6] F. Jaeger, *On nowhere-zero flows in multigraphs*, in *Proceedings of the Fifth British Combinatorial Conference* (C.S.A. Nash-Williams and J. Sheehan, eds.), Congressus Numerantium, No. XV (Utilitas Mathematica Publishing Inc. Winnipeg 1975), 373–378.

[7] F. Jaeger and T. Swart, *Conjecture 1* in *Combinatorics 79* (M. Deza and I.G. Rosenberg, eds.) , Annals of Discrete Math. **9** (North Holland Amsterdam 1980), 305 (Problem Session).

[8] F. Jaeger, *A survey of the cycle double cover conjecture*, in *Cycles in Graphs* (B. Alspach and C.D. Godsil, eds.), Annals of Discrete Math. **27** (North-Holland Amsterdam 1985), 1–12.

[9] P.D. Seymour, *Sums of circuits*, in *Graph Theory and Related Topics* (J.A. Bondy and U.S.R. Murty, eds.) (Academic Press New York 1979), 341–355.

[10] P.D. Seymour, *Nowhere zero 6-flows*, J. Combin. Theory Ser. B **30** (1981), 130–135.

[11] P.D. Seymour, *Nowhere-zero flows*, in *Handbook of Combinatorics vol. 1* (R. Graham, M. Grötschel and L. Lovász, eds.) (North-Holland Amsterdam 1995), 289–299.

[12] J. Shu and C.-Q. Zhang, *Nowhere zero 3-flows in products of graphs*, J. Graph Theory, to appear.

[13] G. Szekeres, *Polyhedral decompositions of cubic graphs*, Bull. Aust. Math. Soc. **8** (1973), 367–387.

[14] W.T. Tutte, *A contribution to the theory of chromatic polynomials*, Can. J. Math. **6** (1954), 80–91.