Comments on the paper "Wallis' sequence ... " by Lampret

Michael D. Hirschhorn

Let

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}.$$

Wallis (1655) showed that $W_\infty = \pi/2$.

In a recent issue of the Gazette [1], Lampret shows that for $n \geq 3$,

$$\frac{\pi}{2}\left(1 - \frac{1.1}{4n}\right) < W_n < \frac{\pi}{2}\left(1 - \frac{0.8}{4n}\right).$$

I would like to go further. We have

$$\frac{W_n}{W_\infty} = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \Big/ \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \prod_{k=n+1}^{\infty} \left(1 - \frac{1}{4k^2}\right).$$

It is easy to check that for $k \geq 2$,

$$\frac{12(k-1)+4}{12(k-1)+7} \Big/ \frac{12k+4}{12k+7} < 1 - \frac{1}{4k^2} < \frac{12(k-1)+5}{12(k-1)+8} \Big/ \frac{12k+5}{12k+8}.$$

If we substitute $k = n+1,\ n+2,\ \ldots$, multiply the results and finally multiply by $\pi/2$, we obtain

$$\frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{7}{3}}\right) < W_n < \frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{8}{3}}\right).$$

which is stronger than Lampret's result.

It can be shown that if the c_k are given by $x\sum_{k\geq 0} c_k x^{2k}/(2k)! = \tanh(x/4)$ then

$$\begin{aligned} W_n &\sim \frac{\pi}{2}\left(1 + \frac{1}{2n}\right)^{-1} \prod_{k\geq 0} \exp\left(\frac{c_k}{n^{2k+1}}\right) \\ &\sim \frac{\pi}{2}\left(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \cdots\right) \text{ as } n \to \infty. \end{aligned}$$

Note that the c_k eventually grow (in magnitude) very rapidly (as $(2k)!/(2\pi)^{2k}$), and the product on the right diverges for every n. However the c_k alternate in sign and for each K,

$$\frac{\pi}{2}\left(1 + \frac{1}{2n}\right)^{-1} \prod_{k=0}^{K} \exp\left(\frac{c_k}{n^{2k+1}}\right) < W_n < \frac{\pi}{2}\left(1 + \frac{1}{2n}\right)^{-1} \prod_{k=0}^{K+1} \exp\left(\frac{c_k}{n^{2k+1}}\right)$$

for n sufficiently large ($n > N(K)$).

References

[1] V. Lampret, *Wallis sequence estimated through the Euler–Maclaurin formula: even from the Wallis product π could be computed fairly accurately*, Aust. Math. Soc. Gazette **31** (2004), 328–339.

Received 25 January 2005, accepted 28 January 2005.