

From tent-like functions to Lucas sequences

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1 Introduction

Counting the number of periodic points of a function leads to results on congruences for the terms of a sequence. Fermat's little theorem (and also its generalization by Euler) can be proved in a simple way, for instance; see [2] and [3]. The paper [5] contains a characterization of the sequences which are *realisable*, that is, sequences which actually appear when counting periodic points of functions. In general, for a given function it can be very difficult or even impossible to compute its iterations and the fixed points of such iterations. But, on the other hand, there are elementary examples, easy to handle by undergraduate students, for which remarkable sequences appear; with this aim we introduce a class of piecewise linear functions for which the number of periodic points is given by the so-called Lucas sequences, well-known to number theorists, and obtain congruence results for those sequences. Modifying this class, the reader will find variants leading to other sequences.

First we give a short account of the abstract theory to place the reader in the appropriate context. Although the study of periodic points is related to the theory of Dynamical Systems, only elementary set-theoretical considerations will play a role here and the metric structure of the domain of the function is completely avoided.

2 Basic terminology and abstract results

Let E be a set and $f : E \rightarrow E$ an function. We say that $b \in E$ is a *fixed point* of f if $f(b) = b$. Define the iterates f^k of f inductively as follows: $f^k(x) = f(f^{k-1}(x))$ for $k \geq 2$, with $f^1 = f$. We say that $b \in E$ is a *periodic point* of f of period n if it is a fixed point for f^n , that is, if $f^n(b) = b$; its *minimal period* is n if moreover $f^j(b) \neq b$ for $j = 1, \dots, n-1$. The fixed points are the periodic points of period 1.

If $f^n(b) = b$, the sequence $\{b, f(b), f^2(b), \dots\}$ (called the *orbit* of b) is periodic in the sense that $f^{k+n}(b) = f^k(b)$. The reader will easily prove the following theorem using elementary divisibility properties of the integers.

Theorem 1 1. *Let b be of minimal period n . Then the orbit of b has exactly n distinct elements, and all of them are of minimal period n .*

2. *If $f^n(b) = b$ and the minimal period of b is k , then k divides n .*

From the first part of the theorem we derive the following corollary.

Corollary 1 *Fix n . If the set $\{x : f^n(x) = x\}$ is finite, then the cardinality of the set of points of minimal period n is a multiple of n .*

From the second part of Theorem 1 we deduce that the set of periodic points of f of minimal period n is given by the solutions of $f^n(x) = x$ which are not solutions of $f^k(x) = x$ for any k divisor of n . According to this observation, if the set of solutions to $f^n(x) = x$ is finite and M_n denotes its cardinality, the number of points with minimal period n can be written as

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) M_d,$$

where μ is the Möbius function, widely used in Number Theory, which is defined as $\mu(k) = (-1)^r$ when k is the product of r different primes, $\mu(k) = 0$ if k has a squared prime factor, and $\mu(1) = 1$.

As a consequence, the result of Corollary 1 can be stated as

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) M_d \equiv 0 \pmod{n}. \quad (1)$$

In particular, if p is prime, $M_p \equiv M_1 \pmod{p}$, and more generally, $M_{p^k} \equiv M_{p^{k-1}} \pmod{p^k}$. If for each integer a we get a function such that $M_n = a^n$, the conclusion $a^p \equiv a \pmod{p}$ for p prime is Fermat's little theorem.

More details appear in [2] and [3].

3 The variants of the tent function

The *tent function* is defined as

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 2 - 2x, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Its graph is formed by two segments ranging from 0 to 1. The graph of f^n consists of 2^n segments ranging also from 0 to 1. It is immediate to deduce that the equation $f^n(x) = x$ has 2^n solutions.

Let \mathcal{T} be the class of continuous piecewise linear functions on $[0, 1]$ whose graph is formed by segments of two types, those ranging from 0 to 1 (the *long legs*) and those ranging from $1/2$ to 1 (the *short legs*); moreover, we require that two legs meet at $x = 1/2$. It is clear that if f belongs to \mathcal{T} , its iterates are also in \mathcal{T} . The tent function is in \mathcal{T} ; it has two long legs and no short legs.

We associate to each $f \in \mathcal{T}$ four numbers, $(A(f), B(f), C(f), D(f))$, which correspond, respectively, to the number of short legs and long legs in the first half of the graph of f (when $x \in [0, 1/2]$), and in the second half (when $x \in [1/2, 1]$).

Theorem 2 *Let $f \in \mathcal{T}$ and write $A = A(f)$, $B = B(f)$, $C = C(f)$, and $D = D(f)$. Let M_n be the number of periodic points of f of period n . M_n is finite for all n except when $C = 1$ and $D = 0$. Excluding this case, and putting $M_0 = 2$, the sequence $\{M_n\}$ satisfies the three-term recurrence relation*

$$M_1 = P, \quad M_{n+2} = PM_{n+1} + QM_n,$$

where $P = B + C + D$, and $Q = AD - BC$.

A sequence with this property is called a *Lucas sequence*. For an account of the definition and properties of Lucas sequences see for instance [6, Section IV, Chapter 2]; for their early history see [1, Chapter XVII, vol. 1]. (We remark that usually the recurrence relation is written with $-Q$ instead of Q .)

Proof. The case $C = 1$ and $D = 0$ corresponds to either $f(x) = x$ or $f^2(x) = x$ for all $x \in [1/2, 1]$.

To prove the theorem we only need to take into account the following elementary facts where, for simplicity, (A_n, B_n, C_n, D_n) are written instead of

$$(A(f^n), B(f^n), C(f^n), D(f^n)):$$

$$(a) \quad M_n = B_n + C_n + D_n;$$

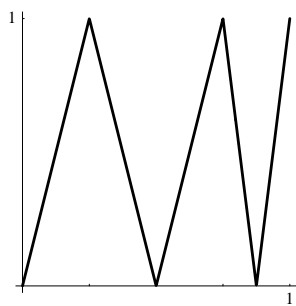
(b) the number of legs of f^{n+1} is given by

$$\begin{aligned} A_{n+1} &= A_n C + B_n(A + C); & B_{n+1} &= A_n D + B_n(B + D); \\ C_{n+1} &= C_n C + D_n(A + C); & D_{n+1} &= C_n D + D_n(B + D). \end{aligned}$$

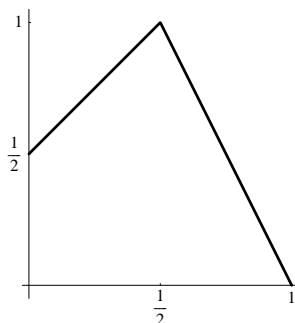
It is enough to check now the values M_1 and M_2 of the sequence and the recurrence relation. □

Remarks:

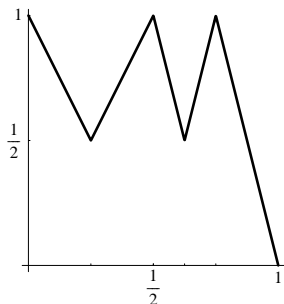
- (i) the case $Q = 0$, which is obtained in particular if there are no short legs, gives the sequence $M_n = P^n$ (Figure 1 corresponds to $P = 5$);
- (ii) the case $P = Q = 1$ corresponds to a Fibonacci sequence; only the choice $A = D = 1$ and $B = C = 0$ gives it (Figure 2);
- (iii) given any pair (P, Q) of positive integers, there is a function in \mathcal{T} , not necessarily unique, for which the corresponding Lucas sequence is obtained (Figure 3 shows the case $P = 3$ and $Q = 2$);
- (iv) if $AD - BC < 0$, a Lucas sequence with negative Q is obtained (Figure 4 gives $P = 3$ and $Q = -1$); nevertheless, one cannot choose a negative value of Q arbitrarily because some sequences are not realisable (for instance, $M_2 \geq M_1$ gives the necessary condition $2Q \geq P - P^2$).



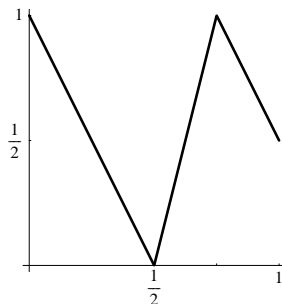
$P = 5$ and $Q = 0$



$P = Q = 1$



$P = 3$ and $Q = 2$



$P = 3$ and $Q = -1$

The case $M_n = P^n$ gives Fermat's little theorem. The Fibonacci sequence $\{1, 3, 4, 7, \dots\}$ is up to constant multiples the only Fibonacci sequence which can be obtained counting periodic points of functions (see [4] and [5]; although the first paper deals with homeomorphisms, this assumption is unnecessary). To obtain multiples of a given sequence paste several copies of a function in \mathcal{T} following the definition $f(x+1) = f(x) + 1$.

When p is prime, the congruence result $M_p \equiv P \pmod{p}$ for the Lucas sequence, which appears in [6, page 41], is now an easy consequence of the general theory. The other congruence results that can be written using (1) do not appear in [6].

References

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