What is Mathematics?

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"So, what's mathematics then?" asked the slightly tipsy man at the bar counter of the one beside him who had foolishly admitted to being a mysterious, legendary creature about whom many fireside tales and seaman's yarn had been woven – a mathematician. The word is only to be whispered in the dark with fear in your eyes and dread in your heart lest a member of that primordial tribe comes to you on fast wings and subjects you to death by boredom with wild tales of conjectures, problems of unspeakable difficulty and – worst of all – analyses of the axiom of choice! The man at the bar has no inkling of the dawn of an idea yet of how ghastly the answer to his imprudent enquiry will be. Take a deep breath, dear reader, for once the threshold of mathematics has been crossed, there is no return. You shall be lured, enthralled and cursed forever with a fascination none but fellows will ever understand.

The present investigator has thrown caution into the wind and with reckless abandon decided to investigate the answer to the question. In the dead of night, hidden in the comforting velvet folds of darkness, questions were put to throngs of mathematicians (7705 individuals and 2339 institutions), 247 of which resolved unwaveringly to answer. Among the questions put to them was this: How would you define 'mathematics'?

This present article is a tribute to the brave people who wrote to me and to all the parents of mathematicians who, with a slightly exasperated look on their faces, finally want to know after all these long and silent years: "What are you doing"? Perhaps then the comments about getting a proper job will cease or begin in earnest. One should not give heed to the fruits of one's labours if one is to be happy in life and so we soldier on, regardless of the outcome. The fact that we could not know the outcome if we did not soldier on shall remain delicately stated in this very sentence. If I had said 'unstated' I would have lied and I have always been taught not to. No matter, let us step once more into the breech, dear friends. (Respondents, who shall remain anonymous, made all statements in double quotes except where credited to someone else.)

1 Works of Reference

"All these difficulties are but consequences of our refusal to see that mathematics cannot be defined without acknowledging its most obvious feature: namely, that it is interesting".
M. Polanyi [1]

Two people pointed their fingers to that hefty tome of modern lore behind which we all hide our ignorance and which we occasionally use to buttress doors — the dictionary. While the set of dictionaries is finite, it is also large in that it contains more than two elements. It has been considered impractical to consult each dictionary. This is mainly due to the assurance of the investigator's friends — and he has no reason to distrust them — that he is not, in fact, capable of reading Chinese and that thus he could not, even if he tried, read all dictionaries. In the dictionaries that one did consult, the number of essentially distinct definitions grew approximately logarithmically with the number of consulted dictionaries. This further substantiates the claim that it is not actually necessary to read all dictionaries. It seems that the writers of dictionaries are prone to the usual human faults of sloth, fraud, plagiarism and eating

too many marshmallows on certain Saturday nights.

A good average over the sample of definitions is provided by the Oxford Dictionary which defines mathematics as "the abstract science of number, quantity, and space studied in its own right (pure mathematics), or as applied to other disciplines such as physics, engineering, etc. (applied mathematics)" [2].

Most definitions wish to allay the reader's fear by fobbing him off with a quick and sweet 'science of numbers'. This woefully inadequate utterance is similar to saying that painting is the art of making vivid colours. While what we have said is part of the collective, it is a small part as, for instance, a certain oxygen molecule in your left nostril is a part of the visible universe.

However there are bigger and better, harder and faster definitions of mathematics around; great Herculean efforts of eloquence that cast the diminutive smurf-like statements we have looked at thus far into the shadow. A case in point or perhaps we should say a point in our case, but this should make no never mind to us since we are not discussing philosophy, is the German dictionary Brockhaus [3] which says:

"The science that originated in the practical problems of calculating and measuring and now concerns itself with the interconnections between quantities and patterns and studies their relationships. This conception was enlarged and deepened by mathematical basic research, i.e. through mathematical logic and set theory. As the science of the structures of interrelations, mathematics gives an overview over all possible, purely logical conclusions given a set of basic assumptions (axioms) where any concepts appearing therein define themselves through the axioms" [4]

Another bulwark is the definition given in the *Collins Dictionary of Mathematics* [5]; it is however just slightly too lengthy to quote here. One respondent came up with a response which is rather typical of that given in dictionaries except that it fails to hit the nail on the head in fewer and better words: "An abstract philosophical language founded on Propositional Logic, based on the notion of 'Mathematical Proof'. Its generic aim is the discovery and demonstration of propositions obtained by combining axioms, definitions and previously proved theorems".

2 The Study of Patterns

"Mathematician's patterns, like the painter's or the poet's, must be beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way".*
G. H. Hardy [6]

The second most prevalent (41 from 247) route of escaping the vice-like hangman's grasp of this gruelling question was to say something such as: Mathematics is "a search for patterns and order in the chaos of life" or, "the science of structures and patterns that bring order and simplicity to our thinking "

At first glance this seems to satisfy us. Virtually all of mathematics is somehow regular and thus contained in this designation. However, we apprehend that it is nonetheless a vacuous statement. By stating 'x is the study of y', we have said nothing of content unless we proceed to detail y. Defining 'pattern', then, is the problem at hand.

As every word in a language is defined in terms of others, this game is a vicious cycle and thus utterly devoid of substance unless we are able to break the circle by a priori knowledge, that is, by a set of understood but undefined terms. It was suggested (by two people), that 'mathematics' should be such a term.

The Concise Oxford Dictionary defines a pattern to be "a regular or logical form, order, or arrangement of parts". In this

way, we attain to the idea that mathematics expresses something which is amenable to meaningful description: A description which highlights features without having to list the details, which draws attention to important general aspects in a veritable sea of information.

The concept of regularity giving rise to predictions is immediately visible in the incantations we choose to call theorems: If x, then y. Given a small amount of information, general statements allow us to sweep clear of further examination of the case at hand but to hold our heads high and to proclaim knowledge with certainty regardless.

3 Mathematics as language, "the unambiguous conversation"

"The mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language".
E. P. Wigner [7]

"Mathematics is both a powerful tool for insight and it's a common language for science. I refer to it as the 'Esperanto' of science".
Rita Colwell [8]

A language is a means for evoking alike thoughts in another person. 17 communicators think of mathematics as a magical art by which one may hurl thoughts across empty space and cause them to germinate in other minds. Oh, who has such power but the mighty mathematician?

Since mathematics is also a study of patterns, the language is based on regularities in the world as well as human thought. This makes mathematics more universal than, for example, English. As it should be, the road to the highest throne must be accessible to all and not a small privileged group of initiates. Anyone can grasp the principles of mathematics by gazing deeply into the forms that shape the world, by listening to the heart and the winds, by tasting life to the fullest and touching the soul of the world. English, in comparison, is more random and arrived at by democracy over thousands of years.

While Esperanto was invented and a total failure in improving cross-cultural interaction, mathematics arises naturally and thus can be said to be discovered. It is highly successful in describing a myriad of things and the face of society has changed in manifold ways since the language of the world has been applied not only to talk about the world but to talk with the world (commonly known as applied mathematics).

"Mathematics is a unique language, in which most of the well-studied and best understood human knowledge can be expressed, stored and communicated with the least amount of loss of information. The reason why it functions seems to be its unequalled simplicity!" It is also "the way of thinking" and consists of "eternal truths that can be communicated over generations".

4 Mathematics as art, "the science of the infinite"

"It's a thing that non-mathematicians don't realize. Mathematics is actually an aesthetic subject almost entirely".
John H. Conway [9]

The eternal debate whether mathematics is discovered or created rages on with unabated ferocity since it was begun thousands of years ago. Old Lore whispers tales of brave and lonely pioneers who duelled in the mists of dawn over their disparate views on whether Plato was right after all to ascribe a perfect, eternal and independent existence in the universe. A responder chose to call mathematics "nature's manual". Here we have a proponent of the ancient ways; indeed most of the members of this lonely brotherhood swear a logicised Hippocratic oath. Since Plato found the cave and every steadfast member

agrees, such mathematicians are called Platonic solids. As we are told by wise men like Lao Tse: In this world, a light side must have a dark side and so there are, hidden on the outer spiral branches of research, some silent rogues who commit unspeakable evil. Dissimilar from the dramatic fall of man, the topsy-turvy and slightly lopsided descent of the mathemagician (read that again) arose not by effecting a choice but by axiomatising it. Hah, we have unearthed your noxious Achilles' heel; take that, accursed fiend! Children are taught early in their lives to ward off such miserable wraiths by chanting Euclid's fifth postulate at them in unison (the light and dark sides run parallel). Elementary school teachers wake up drenched in sweat from their nightmares when – in their dream – another child's faith in the seemingly impregnable walls of geometry has been shattered by the powerful counter curse: "Lobachevsky!"

Conversely, some strike boldly out into the public eye and demand that it look at them with an enduring capable eye and acknowledge that they are "strongly opposed to all flavours of Platonism and side with a version of formalism with certain pragmatic overtones". These noble princes of the other side are brave and valiant and must be honoured.

While mathematicians' salaries prominently illustrate the popular maxim 'art for art's sake', there are many who actually subscribe to this high ideal of Platonic love to the other solids (don't ask!). So, while some might say "combinatorial intellectual art", others quote Weierstrass: "A mathematician who is not also a little bit of a poet is not a good mathematician". Even though only eight responders explicitly mentioned the connection with art, the view seems widespread that mathematics involves a creative impulse. Whatever it is, we are doing it "for the fun of it".

We seek "purest beauty in a human-defined universe of logical structures" and we "feel 'mathematics' as a mental outburst of an animate" being. Mathematics is "just as difficult as defining 'Brahman': not like this, not like this" and it is also "the will for uniqueness". Mathematicians also share the artistic spirit of "curiosity, inductive reasoning, intuition, [and] logic or deductive reasoning".

5 Mathematics as logic

"Mathematics is the science which draws necessary conclusions".
Benjamin Peirce [10]

42 people felt the logical aspect of mathematics as very important. It is believed that this count may have a touch to do with the Great Question of life, the universe and everything but since any sane even-tempered teetotaller cannot regard this to have even a shimmer to do with logic, this must be a titanic mound of dingo's kidneys! The antediluvian directive, 'thou shalt be logical' echoes still in the fabric of the earth; you can hear it if you put your ear to the ground on a hushed and desolate night illuminated by a full moon.

One might say that "within mathematics you can include everything that depends entirely and only on logical reasoning" and mathematics is the "study of well-defined things". It is as clear as an alpine pool of pristine, fresh and cool mountain water and is a definition intended to appease the fears of the layman whilst heightening his feeling of prestige for the mighty warriors of the mind that are the mathematicians. The lulling tale of logic is enchanting to the mind as it provides no understanding but creates the perception of it. We may quote a marvellous book [11] that should be required reading for every apprentice and master mathematician: "Mathematicians always strive to confuse their audiences; where there is no confusion there is no prestige. Mathematics is prestidigitation. It is really the man who is totally at sea who has got both feet firmly on the ground".

In spite of the primordial drive to confuse the uninitiated and to lock ourselves up in that beautiful ivory tower of impregnable logic, there are some weaknesses that we must confess at least to ourselves when we have returned from the battle fields of conferences: "Mathematics is the compulsion to define every problem very precisely".

6 The pursuit of happiness through numbers"

"A scientist worthy of the name, above all a mathematician, experiences in his work the same impression as an artist; his pleasure is as great and of the same nature".
Henri Poincaré

Nothing is a nobler quest than that of happiness — happiness, that is, of the deep kind; a loving, universal, contented and wise sort of happiness that cheers the world, makes flowers bloom, chases the cold winter away and generally evokes joy in all who come in contact with the exultant one. There are many roads to Rome but only one is the road of pilgrimage — the geodesic to divine agape. Can one find happiness by contemplating the eternal beauties of numbers? Who are you and I to say 'nay' when we have not achieved such an exalted state by any means at all? Perhaps the "science about the numbers and everything that derives from there" is the king's highway or perhaps just a dingy old alleyway but if it does the trick, shall we waver?

Bunching all the pilgrims of mathematics into the narrow road of the real line is surely cramped albeit an injection is clearly possible. Yet there are 30 of our valiant knights who regard "the study of numbers" as the royal road. Godspeed them!

One may glorify and expand the game of numbers by saying: "Mathematics is the science of numbers. Mathematics explores every aspect of numbers, how we use them, how they are applied to day-to-day life and how they are applied to science".

7 Modelling the physical universe with the "Queen of all sciences"

"Mathematics is the alphabet in which God described the universe".
Galileo Galilei

The physical universe, the cosmos in which our sun and our planets dance to the ancient music of the spheres and chant the primeval mantra 'aum', is the inspiration of all thinkers. Many are contented in contemplating whether the universe is real, a God exists and what the purpose of human life might be. 23 of our lot are willing to suspend their disbelief (or not to come to any belief about these issues at all) and only wish to discover the past and reveal the future; they desire the chronicles of Akasha and long to read "Nature's manual".

The natural sciences are concerned with predicting the future state of physical systems given their present state. As a matter of practicality we wish to know the precise moment (to five decimal places) of when a little boy's bicycle will break. We resolve in the good spirit of physics that we must never ask why but are allowed to ask how, please, it is to be fixed. Dealing with the broken bicycle, the crying boy, the injured knee and the ruined rose bushes of the elderly McTempers is going to be nasty. Thus, there can be no surprise that predicting the future state of systems is called a 'problem'. Mathematics might then be called "the fine art of problem solving". Now a fine art indeed it is. One approach would be to slowly read Andrew Wiles' proof of Fermat's Last Theorem to everyone which would solve almost everything as both the McTempers and the boy would be quietly snoring away after the abstract, leaving the bike and the roses which reduces the given problem to previously solved ones — it is common practise in mathematics, of course, to stop there and let the common masses do the dirty work of actually mending things.

This is a prime example of how mathematics may be used indirectly to solve a physical problem. But are there direct ways of wielding Excalibur mathematically? Ha, oh ye of little faith, mathematics is "the way to model life". We hasten to add that 'life' must be taken with a pinch of salt here (you need the sodium), as we do not intend to be able to model your wedding night very precisely, not that we want to, but anyhow we speak merely of the physical and biological aspects on a small scale. In this way, mathematics becomes a tool (like tealeaves and such) in the great endeavour of foretelling the misty future accurately. Astounding revelations that will shatter your conception of reality can be made. They will leave you so mind bogglingly speechless that, in fact, a good deal of it is classified by the more militant members of the government under the pretence of your own safety by which they mean their own safety: On the basis of the theories of gravity and the solid state, for example, we may say with reasonable confidence that if a delicate glass is dropped from a tall building, it will break. Are you not gob-smacked and ogling the next mathematics book in sight of this prophetic proclamation?

Another contemptuous view of the greatness of mathematics is to cast it down even lower than the lofty status of tealeaves in the prediction business and to call it "a means to teaching physics". Predicting things depends on how they are now and prediction is the supreme goal of the natural sciences. From their view of mathematics as a tool to help them one may safely say, "everything, sooner or later, comes down to measurement". If the present is fuzzy, the future is blank as anyone who takes pleasure in alcohol can attest to.

8 What does a mathematician do?

"A mathematician is a human who will not only immediately understand any thought described to him but will also see which fallacy of thought it rests upon".
Helmar Nahr

We relentlessly pursue the tracks of that slippery and devious creature which seeks to confound its hunters. In the sweet morning dew we lie hidden underneath our camouflage of discarded scribbles and observe the rare ritual of the mathematicians mating call: 'I think I've proved your theorem!' This magical incantation has a truly scandalous flabbergasting magnetism on the mathematician addressed therein. Irresistibly, the conjecturer is drawn in a delicate balance of many emotions such as disbelief, anger, surprise, jealousy, interest and envy.

As the mathematician is such a complex beast, eight people prefer to define mathematics as "that which mathematicians do". Again we hasten to add that this does not refer to anything they might do after hours beneath the sheets but rather in their minds and on their pads of recycled paper. Mathematicians build "huge buildings, some beautiful, some useful, resting safely on tiny foundations and with almost zero costs". These master masons are justly wondrous, they have created castles with towers and turrets of such beauty and breathtaking elegance that the Taj Mahal would crumble in shame at their mere mention (that is why there are no conferences held there). Even Herodotus could tell us of no greater wonder in the world than that of the buildings of thought which have been shaped by scores of skilful minds for over three thousand years. Mathematicians follow "exact rules to the utmost accuracy and all that with brilliant imagination". And so the palace stands, untouched by the elements, ever increasing in their splendour through the tireless efforts of the mathematicians. It is through thought alone that the past efforts are preserved and the present ones leave their mark forever, so that "the process of mathematics is the organised investigation of patterns derived from axioms;

the body of mathematics is the collected remembrance of those investigations".

9 The amorphous subject

"Mathematics has a fractal structure: If you look closer and closer to its subjects, more and more structures evolve". From this point of view, three mathematicians liken mathematics to a closed Koch curve: it enclosed a finite area but has an infinite (and fuzzy) boundary. There is much about which one is uncertain whether it is to be included or not. In contrast to our best mathematical intuitions about logic, proof, truth and such, sometimes we must harden our hearts, steel our eyes, sharpen our pencils and decide against a murky 'yes' or and cloudy 'no' but exclaim loud and proud with a clear, crisp and totally useless 'perhaps'.

If you are inclined to dispute even the rock solid foundations of the upside-down pyramid of mathematics you might press your chisel against a brick in the masonry and say that a closed curve such as the closed Koch curve does not, in fact, divide the plane into two bits (Jordan's curve theorem) and proclaim that "everything is mathematics, it's just that people aren't aware of it".

10 Towards a new definition

Mathematics is "giving the same name to different things".
Henri Poincaré

To recap on the popularities of the different avenues of resolution two mentioned the dictionary, 41 patterns, 17 language, 8 art, 42 logic, 30 numbers, 23 modelling, 8 what mathematicians do and three the fuzziness.

We are faced with the final challenge. At the end of the road, we must swim the moat and we shall be home in our bastion at last. "One of the great achievements of mankind" must receive an identity, but how? We may well envisage that "the definition would not contain the word 'number', and it would say something about the artistic aspect of math". Nonetheless 52 of our mighty warriors succumbed on the weary road and did not answer the question.

Having walked shadowy chasms of dizzying depths and fearlessly inquired about the "subject with no object", I humbly present for your perusal a definition which represents, to the best of my estimation, an acceptable compromise between existing definitions, the 247 responses received and that elusive, legendary concept — truth:

Math.e.mat.ics *n. pl.* [from the Greek *mathema* "to learn" which came from the Sanskrit *medha* "wisdom and intelligence"] A collection of subjects that investigate particular kinds of patterns that are derived from the physical universe, abstract thought or the imagination with the aim of giving logically correct analyses of the properties of these patterns. Examples of the subjects that make up mathematics are: Geometry (deals with points and lines in any kind of space), Topology (considers shape of objects in space irrespective of their size), Algebra (examines the relationships of symbols given certain assumed properties of these symbols), Analysis (investigates functions and continuity), Number Theory (studies the abstract properties of numbers) as well as Logic and Set Theory. Mathematics is typically divided into pure and applied mathematics, a distinction that applies to the practise rather than to the subject. If mathematics is practised with a benefit to subjects outside of mathematics in mind (other sciences, society, industry, etc.), then it is applied mathematics; if it is practised for its own sake, it is pure mathematics. While the record of the constituting subjects is in the form of short statements about the properties of the investigated patterns (theorems) and the logically consistent justification of these statements (proofs), the process of arriving at this linear, logical and scientific record is significantly different. The practitioner of mathematics (mathematician) proceeds by using a keen sense of beauty and elegance of the

pattern to be studied, intuition about what properties the pattern could have and how one might prove it. As an evolving domain of learning, mathematics is thus as much an artistic discipline as a science. Mathematics is also a language in which any kind of regularity can be conveniently expressed and, for that reason, it is the language of natural science and engineering and quickly becoming an indispensable part of the language of social, economic and other fields of study. The collection is so varied and vast that it is frequently uncertain whether a given factoid is to be included under the umbrella that is mathematics.

11 The questionnaire and the book

As mentioned in the preamble, many more questions were asked: Questions about personal experiences, fascinations, decisions along the road, the community of mathematicians, the gender problem and written works are contained, among others, in the questionnaire. Together with Nicky Graves-Gregory, I am writing a book on what mathematicians are like as people, how their lives were shaped by mathematics and how they moulded it. We are interviewing eight prominent mathematicians and have conducted this questionnaire to represent those who we did not have time to interview in person. The analysis and presentation of the rest of the questionnaire together with eight brilliant interviews will be contained in that book which is currently in progress.

12 Acknowledgements

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