More terms of the asymptotic solution of a difference equation considered by Ramanujan

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Hirschhorn [1] (following Ramanujan) studied the difference equation

$$u(\lambda) + \frac{1}{u(\lambda-1)} = 1 + e^{\lambda x}$$

and expanded it with the help of Maple up to the powers of x^2. He then writes: *"It is tempting to go on and try to extend the series to the term in x^3, but life isn't long enough!"*

We make the *ansatz*[1]

$$u(\lambda) = \frac{\lambda + C + 1}{\lambda + C} + \frac{a(\lambda)x}{(\lambda+C)^2} + \frac{b(\lambda)x^2}{(\lambda+C)^2} + \frac{c(\lambda)x^3}{(\lambda+C)^2} + \frac{d(\lambda)x^4}{(\lambda+C)^2} + \frac{e(\lambda)x^5}{(\lambda+C)^2} + \frac{f(\lambda)x^6}{(\lambda+C)^2} + \frac{g(\lambda)x^7}{(\lambda+C)^2} + \frac{h(\lambda)x^8}{(\lambda+C)^2} + \cdots$$

and could compute all the coefficients up to powers of x^8 within a few minutes of Maple's time.

After a suitable change of variable, there was always a recursion

$$t(i) - t(i-1) = \text{SOMETHING}(i),$$

for $t(i) = a(i), b(i), \dots, h(i)$, from which we conclude that

$$t(i) = \sum_{j=1}^{i} \text{SOMETHING}(j) + t_0;$$

in all instances Maple could easily produce a closed form for the sum involved.

As an example, we cite

$$\begin{aligned} b(i) - b(i-1) = &-\tfrac{1}{16}i^6 - \tfrac{1}{48}(10C-9)i^5 - \tfrac{1}{144}(31C^2 - 45 - 75C)i^4 \\ &- \tfrac{1}{144}(-62C^2 + 4C^3 - 9 - 84C)i^3 + \tfrac{1}{144}(72a_0 + 6C^3 + 3C^4 + 15C + 40C^2)i^2 \\ &- \tfrac{1}{144}(3C^4 - 48Ca_0 + 72a_0 + 2C^5 - C^2)i - \tfrac{1}{6}Ca_0 - \tfrac{1}{6}C^2a_0 + \tfrac{1}{144}C^5 + \tfrac{1}{144}C^6 \\ &- \tfrac{1}{144}C^3 - \tfrac{1}{144}C^4 + \tfrac{1}{144}\tfrac{(-12a_0 - C^2 + C^4)^2}{i+C} - \tfrac{1}{144}\tfrac{(-12a_0 - C^2 + C^4)^2}{i-1+C}. \end{aligned}$$

So we obtained (C, a_0, ..., h_0 are unspecified constants)

$$a(\lambda) = \frac{2\lambda^3 C}{3} + \lambda^2 C + \frac{\lambda C}{3} + \frac{\lambda^2 C^2}{2} + \frac{\lambda C^2}{2} - a_0 + \frac{\lambda^3}{2} + \frac{\lambda^2}{4} + \frac{\lambda^4}{4},$$

$$\begin{aligned} b(\lambda) = -\frac{1}{5040(\lambda+C)C}\Big[&-1260\lambda^5 C - 735\lambda^4 C + 5040{a_0}^2\lambda + 220C^2\lambda^7 + 392C^3\lambda^6 + 252C^4\lambda^5 + 45C\lambda^8 \\ &- 1820\lambda^3C^2 - 3780\lambda^3C^3 - 1372\lambda^2C^3 - 1260\lambda^2C^4 + 60\lambda^2C + 60\lambda C^2 + 5040b_0C\lambda \\ &- 840Ca_0\lambda^4 - 1680C^2a_0\lambda^3 - 630\lambda^6C - 2240\lambda^5C^2 - 2800\lambda^4C^3 + 1680\lambda C^2a_0 \\ &- 1260\lambda^3C^4 + 840\lambda^2Ca_0 + 5040b_0C^2 - 252\lambda C^4 - 3780\lambda^4C^2\Big], \end{aligned}$$

[1] Slightly different from Hirschhorn, but that seems to work best for Maple

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\begin{aligned}
c(\lambda) = -\frac{1}{181440C^2(\lambda+C)^2}\Big[& -1215\lambda^{10}C^2 - 6900\lambda^9C^3 - 15492\lambda^8C^4 - 16296\lambda^7C^5 + 60480\lambda^4C^5 - 90720\lambda^2C^3b_0 \\
& - 75600\lambda^2C^2a_0^2 + 35280\lambda^4C^4a_0 - 22176\lambda^2C^4a_0 - 60480\lambda C^4b_0 + 15120C^2\lambda^7 \\
& + 60480C^3\lambda^6 + 90720C^4\lambda^5 - 2088\lambda^3C^3 - 1224\lambda^2C^4 - 13860C^2a_0\lambda^4 + 30240Ca_0^2\lambda^5 \\
& + 75600C^2a_0^2\lambda^4 + 30240\lambda^5b_0C^2 + 90720\lambda^4b_0C^3 + 60480\lambda^3C^4b_0 - 3060a_0C^2\lambda^8 + 3024\lambda^2C^6 \\
& - 181440a_0^3\lambda^2 + 81\lambda^{12}C^2 + 572\lambda^{11}C^3 + 1618\lambda^{10}C^4 - 181440c_0C^2\lambda^2 - 362880c_0C^3\lambda \\
& - 12384a_0C^3\lambda^7 - 13104a_0C^4\lambda^6 - 362880a_0b_0C\lambda^2 - 32256C^3a_0\lambda^3 + 2156\lambda^9C^5 + 1134\lambda^8C^6 \\
& + 12600\lambda^6C^2a_0 - 30240\lambda^3b_0C^2 - 30240\lambda^3Ca_0^2 + 40320\lambda^5C^3a_0 + 4320a_0\lambda^2C^2 \\
& - 181440c_0C^4 - 362880a_0b_0C^2\lambda - 6804\lambda^6C^6 + 17766\lambda^4C^6 + 8235\lambda^6C^2 + 4320\lambda C^3a_0 \\
& + 18256\lambda^3C^5 + 28660\lambda^5C^3 + 35552\lambda^4C^4 + 15120\lambda^3C^6 + 8883C^2\lambda^8 + 70266C^4\lambda^6 \\
& + 56364C^5\lambda^5 + 40236C^3\lambda^7 - 864\lambda^4C^2\Big],
\end{aligned}
$$

$$
\begin{aligned}
d(\lambda) &= \frac{1}{C^3(\lambda+C)^3}[\text{123 terms}], \\
e(\lambda) &= \frac{1}{130767436800C^4(\lambda+C)^4}[\text{242 terms}], \\
f(\lambda) &= \frac{1}{53219731428864000C^5(\lambda+C)^5}[\text{474 terms}], \\
g(\lambda) &= \frac{1}{638636777146368000C^6(\lambda+C)^6}[\text{855 terms}], \\
h(\lambda) &= \frac{1}{882200071214449827840000C^7(\lambda+C)^7}[\text{1510 terms}].
\end{aligned}
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References

[1] M. D. Hirschhorn, *The asymptotic solution of a difference equation considered by Ramanujan*, AustMS Gazette **31** (2004), 120–124.

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