The 8th Problem

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In the year 2000, exactly one hundred years after David Hilbert posed his now famous list of 23 open problems, The Clay Mathematics Institute (CMI) announced its seven Millennium Problems. (http://www.claymath.org/millennium). Any person to first publish a correct solution, proof or disproof of one of the following problems: 1) Birch and Swinnerton–Dyer Conjecture, 2) Hodge Conjecture 3) Navier–Stokes Equations 4) P versus NP 5) Poincaré Conjecture 6) Riemann Hypothesis 7) Yang–Mills Theory, does not only earn immortal fame but will be awarded the generous sum of one million US dollars. With Perelman's (likely) proof of the Poincaré Conjecture, the continued optimism about an impending proof of the Riemann Hypothesis, and the omission of such famous problems as Twin Primes and Goldbach, it seems the CMI would have been wise to have followed Hilbert's example in announcing not 7 but 23 Millennium Problems. In the coming months the Gazette will try to repair the situation, and has asked leading Australian mathematicians to put forth their own favourite 'Millennium Problem'. Due to the Gazette's limited budget, we are unfortunately not in a position to back these up with seven-figure prize monies, and have decided on the more modest 10 Australian dollars instead.
In this issue Alexander Molev formulates what from now on should be referred to as Millennium Problem number eight 'Schubert polynomials and the Littlewood Richardson rule'.

Littlewood–Richardson problem for Schubert polynomials

This note is my response to the editors' request to write about my favourite open problem which could have made it to the CMI Millennium Problem list. I have chosen to discuss the multiplication rule for the Schubert polynomials.

The symmetric group $\mathfrak{S}_n$ acts on polynomials in $x_1, \dots, x_n$ by permuting the variables. Given a polynomial P, the difference $P - s_i\, P$ with $s_i = (i, i+1)$ is antisymmetric in x_i and x_{i+1}, hence divisible by $x_i - x_{i+1}$. Define the *divided difference operator* ∂_i by

$$\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i).$$

These operators satisfy

$$\begin{aligned} \partial_i^2 &= 0, \\ \partial_i\, \partial_j &= \partial_j\, \partial_i \qquad \text{if} \quad |i-j| > 1, \\ \partial_i\, \partial_j\, \partial_i &= \partial_j\, \partial_i\, \partial_j \qquad \text{if} \quad |i-j| = 1. \end{aligned}$$

Hence, if $w = s_{i_1} \cdots s_{i_p}$ is a decomposition of $w \in \mathfrak{S}_n$ with minimal length then we can unambiguously set $\partial_w = \partial_{i_1} \cdots \partial_{i_p}$. For any $w \in \mathfrak{S}_n$, Lascoux and Schützenberger [3] defined the *Schubert polynomial* $\mathfrak{S}_w$ by

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}),$$

where the permutation w_0 is defined by $w_0(j) = n - j + 1$. The polynomials $\mathfrak{S}_w$ turned out to possess a number of remarkable properties and they have since been studied by many authors from the algebraic, combinatorial and geometric viewpoints. An exposition of this beautiful theory is found in the monographs by Macdonald [5] and Manivel [6]. It can be shown that $\mathfrak{S}_w$ is independent of n (when $w \in \mathcal{S}_n$) and the 'first' few of the $\mathfrak{S}_w$'s are

$$\mathfrak{S}_{\mathrm{id}} = 1, \quad \mathfrak{S}_{s_1} = x_1, \quad \mathfrak{S}_{s_2} = x_1 + x_2, \quad \dots, \quad \mathfrak{S}_{s_k} = x_1 + \dots + x_k,$$
$$\mathfrak{S}_{s_1 s_2} = x_1 x_2, \quad \mathfrak{S}_{s_2 s_1} = x_1^2, \quad \mathfrak{S}_{s_1 s_2 s_1} = x_1^2 x_2, \quad \mathfrak{S}_{s_1 s_3} = x_1^2 + x_1 x_2 + x_1 x_3.$$

The polynomials $\mathfrak{S}_w$ with w running over all finite permutations of the set of positive integers form a $\mathbb{Z}$-basis of the ring $\mathbb{Z}[x_1, x_2, \dots]$. The product of two Schubert polynomials can therefore be expanded as

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{uv}^w \, \mathfrak{S}_w, \tag{1}$$

where the integers c_{uv}^w are known to be nonnegative. This leads to what many combinatorialists consider to be the most important open problem in the theory of Schubert polynomials.

Littlewood–Richardson Problem. *Find a combinatorial rule for the determination of the coefficients* c_{uv}^w.

A permutation w is called *Grassmannian of descent* k if $w(k) > w(k+1)$ and for any $j \neq k$ we have $w(j) < w(j+1)$. Any such permutation defines a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_{k+1-j} = w(j) - j$. The corresponding Schubert polynomial $\mathfrak{S}_w$ coincides with the *Schur polynomial* $s_\lambda(x_1, \dots, x_k)$ which can be defined as the ratio of two determinants,

$$s_\lambda(x_1, \dots, x_k) = \frac{\det[x_j^{\lambda_i + k - i}]_{i,j=1}^k}{\det[x_j^{k-i}]_{i,j=1}^k}.$$

If k is fixed and λ runs over all partitions having at most k parts, then the corresponding Schur polynomials form a distinguished $\mathbb{Z}$-basis in the ring $\mathbb{Z}[x_1, \dots, x_k]^{\mathcal{S}_k}$ of symmetric polynomials. The coefficients $c_{\lambda\mu}^\nu$ in the expansion

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu \, s_\nu$$

are calculated by the classical *Littlewood–Richardson rule* [4]. A combinatorial rule for the calculation of the coefficients c_{uv}^w for the product of two Schubert polynomials would thus generalize the Littlewood–Richardson rule. Some particular families of the coefficients c_{uv}^w were calculated e.g. in Bergeron and Sottile [1] and Kogan [2].

The significance of the Littlewood–Richardson problem for the Schubert polynomials is explained by its close relationship with the geometry of the flag varieties. Let $V = \mathbb{C}^n$. A *flag* in V is a sequence of subspaces

$$\{0\} = F_0 \subset F_1 \subset \dots \subset F_{n-1} \subset F_n = V$$

with $\dim F_i = i$. The set $\mathbb{F}_n$ of all flags is a complex algebraic variety of dimension $n(n-1)/2$. By a theorem of Borel, the cohomology ring $H^*(\mathbb{F}_n)$ of $\mathbb{F}_n$ is isomorphic to the quotient of the ring of polynomials

$$H^*(\mathbb{F}_n) \cong \mathbb{Z}[x_1, \dots, x_n]/I, \tag{2}$$

where I is the ideal of $\mathbb{Z}[x_1, \dots, x_n]$ generated by all elementary symmetric polynomials in $x_1, \dots, x_n$. The images of the Schubert polynomials $\mathfrak{S}_w$ with $w \in \mathcal{S}_n$ in the quotient ring $\mathbb{Z}[x_1, \dots, x_n]/I$ form a $\mathbb{Z}$-basis of that ring. These images correspond to distinguished

Schubert classes $\sigma_w \in H^*(\mathbb{F}_n)$ under the isomorphism (2). Then for any $u, v \in \mathcal{S}_n$ in the cohomology ring we have

$$\sigma_u \cdot \sigma_v = \sum_{w \in \mathcal{S}_n} c_{uv}^w \, \sigma_w,$$

where the c_{uv}^w are defined in (1). Thus, the coefficients c_{uv}^w determine the ring structure of $H^*(\mathbb{F}_n)$. They may be interpreted as certain intersection multiplicities and are therefore nonnegative integers—something that does not yet have a combinatorial proof!

References

[1] N. Bergeron and F. Sottile, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J. **95** (1998), 373–423.

[2] M. Kogan, *RC-graphs and a generalized Littlewood-Richardson rule*, Internat. Math. Res. Notices **15** (2001), 765–782.

[3] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris **294** (1982), 447–450.

[4] D.E. Littlewood and A.R. Richardson, *Group characters and algebra*, Philos. Trans. Roy. Soc. London **233** (1934), 99–142.

[5] I.G. Macdonald, *Notes on Schubert polynomials*, Laboratoire de combinatoire et d'informatique mathématique, Université du Québec à Montréal, 1991.

[6] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, Vol. 6 (AMS Providence 1998).

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